Journal of Geometry and Physics 11 (1993) 15–62
North-Holland

JOURNAL OF
GEOMETRY AND
PHYSICS

Aspects of canonical gravity and supergravity

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In these lectures, we review some recent developments in canonical gravity and supergravity with special emphasis on issues related to Ashtekar's variables. Their construction and the formal solutions to the quantum constraints of pure gravity in four dimensions are discussed at an introductory level. We then consider topological ($N = 1$) and matter coupled ($N = 2$) supergravity in three dimensions. For $N = 1$ supergravity we derive the observables and a complete set of solutions to the quantum constraints. Finally, we work out the canonical structure of $N = 2$ supergravity and show that there exist physical observables based on "hidden symmetries". The quantization of this theory is briefly discussed.

Keywords: *canonical gravity, supergravity*
1991 MSC: 83 E 50, 83 C 45

Contents

1. Introduction and motivation

The present lectures review some recent developments in canonical classical and quantum gravity and supergravity at an introductory level, with special emphasis on issues related to Ashtekar's variables. The discovery of these variables is widely regarded as one of the significant advances of the last few years and has greatly stimulated recent research in canonical gravity. Since there are already several excellent introductory texts available (see, e.g., ref. [1] and the references listed there for a review of the general theory within the more familiar metric formalism and ref. [2] for a discussion of the new variables), special emphasis will here be placed on some topics that have not received so much attention in the existing literature.

The difficulties that one is confronted with in searching for a consistent quantum theory of gravity appear in several guises. First and foremost, there are enormous conceptual problems in understanding "what really happens" as one approaches the Planck scale where the conventional notion of space–time as a differentiable manifold must necessarily break down. Secondly, there are severe technical problems, most notably the fact that gravity, when viewed as a conventional quantum field theory, is non-renormalizable and thus in need of modification at short distances. Present attempts to come to grips with these problems center around three different approaches. Among these, the most "physical" proceeds by gedanken experiments (as well as a certain amount of "gedanken theory") to probe physics at the Planck scale. For instance, one studies high (i.e. Planckian) energy scattering of elementary particles and strings (see, e.g., ref. [3]), or tries to unlock the secret of quantum gravity through a better understanding of black hole physics [4,5]. The advantage of this approach is that it relies on physical intuition rather than formal mathematics; however, it leaves aside questions of mathematical consistency. The approach most popular with particle physicists concentrates on the perturbative structure of the theory. Here one tries to cure the short distance singularities of quantum gravity by adding suitable matter to cancel the infinities and thereby to arrive at a mathematically consistent theory. This approach has so far led to supergravity [6] and superstrings [7], the first theory where all divergences are allegedly absent. Unfortunately, little has been learnt about the conceptual difficulties of quantum gravity until now by following this route, but one hopes that the undoubtedly beautiful mathematical structures of these theories might ultimately lead to some better understanding of the conceptual issues as well. If successful, this approach could possibly explain the spectrum of elementary particles from the postulated absence of divergences.

Finally, one can apply canonical quantization methods to gravity [8–10]. This is, in a sense, the most conservative approach since, at least at the initial stage, only a good knowledge of textbook methods is required besides knowledge of Einstein's theory. This approach appears to be well suited for the investigation of the conceptual problems of quantum gravity, and has also led to some interesting formal developments. On the other hand, it has little to say about the origin of elementary particles (in particular fermions); their explanation is usually viewed

Fig. 1. Interpreting the wave function of the Universe [11].

as a secondary problem in the canonical framework and left to other approaches to solve.

In these lectures, we will focus on the canonical approach with a special eye towards a particle physics audience. The emphasis here will be on the technical rather than on conceptual issues. So we will have nothing to say about the possible interpretation of the wave function of the universe; see, however, fig. 1 for some suggestions. We will thus simply apply the usual quantization prescriptions (supplemented by Dirac's theory of constrained Hamiltonian systems [8]) to Einstein's theory. This procedure leads to a Schrödinger type equation which is commonly referred to as the "Wheeler–DeWitt equation" [12,13]. Of course, this equation is far more complicated than an ordinary Schrödinger equation, and attempts to find genuine and physically meaningful solutions (so-called "wave functions of the universe") have largely failed. The difficulties are particularly acute in the familiar metric formalism, where one ends up with a highly non-linear functional differential equation, which is practically impossible to solve as it stands. In order to break the deadlock, one can try to mutilate it by retaining only a finite number of degrees of freedom (so-called "mini superspace approximation"), but one cannot hope to get more than a caricature of the real world in this way.

An important breakthrough occurred in 1986 with the discovery of new phase space variables in terms of which the canonical constraints become *polynomial* [14,15] (it is a measure of the complexity of Einstein's theory that it took more than seventy years for this discovery!). With the new variables, it becomes possible to construct non-trivial (though formal) solutions to all the constraints [16,17]. The fact that the interpretation of these solutions remains obscure can be viewed as an indication of the highly unfamiliar features of quantum gravity. Of course, one has no right to expect that a mere change of variables will be sufficient to solve all the problems of quantum gravity, but the formalism contains sufficiently many new and promising elements to justify some optimism. This is especially true when one combines it with other concepts and ideas such as supergravity and superstrings. We believe that further progress will depend on giving up isolationist viewpoints, and that interesting developments in the near future may well occur on the interface between canonical gravity and superstring theory.

Among the new topics to be treated here are three-dimensional supergravities, where solutions of the quantum constraints of the $N = 1$ theory are explicitly derived for the first time [#1]. Furthermore, we will discuss "hidden symmetries" in the canonical framework and their relevance for the construction of observables in the sense of Dirac. $N = 2$ supergravity will be treated in quite some detail here as it is the simplest matter coupled theory exhibiting such symmetries and thus provides a simpler example of the canonical treatment of extended supergravity than the $N = 16$ theory discussed in ref. [18]. Although our results are far from complete, we also briefly consider the quantization of the $N = 2$ theory. We hope that our arguments will convince the reader that, despite the numerous open problems, dimensionally reduced supergravities are especially interesting theoretical laboratories to enlarge the scope of the formalism to matter coupled theories and perhaps to generalize it to higher dimensions, following suggestions of ref. [18].

We now summarize some notations and conventions used in these lectures. We will use capital letters $M, N, \ldots$ and $A, B, \ldots$ to label curved and flat indices, respectively, in d dimensions. Similarly, indices $m, n, \ldots$ and $a, b, \ldots$ will be employed to label tensors in $d - 1$ (spatial) dimensions. We will use the special indices t and 0 for the curved and flat time component, so M (A) takes the "values" m and t (a and 0). A dot or ∂_t will denote time derivatives. The metric has signature $(- + \cdots +)$ in d dimensions; γ matrices obey $\{\gamma^A, \gamma^B\} = 2\eta^{AB}$.

2. Basics of canonical gravity

The canonical treatment of gauge theories relies on the methods developed by Dirac in his study of constrained systems [8] (see also refs. [19,20]). Local invariances in gauge theories imply constraints. The most elementary example is is ordinary electrodynamics, where the absence of $\partial_t A_t$ from the Lagrangian implies that the associated canonical momentum vanishes. Consistency then

[#1] This part of the lectures is based on unpublished joint work with I. McArthur. We are grateful to him for the permission to include these results here.

requires that this constraint is preserved by the time evolution, which in turn leads to Gauss' law $\partial_m E^m = 0$. More generally, suppose that ϕ is a field without time derivative in the Lagrangian $\mathcal{L}$. Then we find a primary constraint $\Pi = \delta\mathcal{L}/\delta\dot{\phi} = 0$, and we must have

$$\dot{\Pi} = (\delta\mathcal{L}/\delta\phi)\,\{\phi,\Pi\} = 0\,, \tag{2.1}$$

which requires

$$\delta\mathcal{L}/\delta\phi \approx 0\,, \tag{2.2}$$

where the left hand side of (2.2) is to be expressed in terms of the canonical variables (this is called a secondary constraint). The above equation already makes use of standard notation: "≈ 0" means "weakly zero", i.e., the constraints must be imposed only *after* all canonical brackets have been calculated [8]. Constraints such as (2.1) or (2.2) single out a hypersurface in the phase space of the theory. In the quantum theory, the constraint must be imposed as an operator constraint on the Hilbert space: the states selected in this way are called "physical states". We will not always distinguish between classical and quantum theory. So, *par abus de langage*, the word "commutator" will refer to both the classical (Poisson or Dirac) bracket and the quantum commutator.

In this section, we will review the application of Dirac's formalism to gravity. The basic steps here are, of course, well known [9,10], but we nonetheless present some details, not only to set up the notation, but also to make the presentation reasonably self-contained. One starts by slicing ("foliating") space–time into a sequence of space-like hypersurfaces; this step violates the manifest invariance under four-dimensional general coordinate transformations (or diffeomorphisms in more mathematical parlance). The configuration space variables of gravity are the ten components of the metric tensor $g_{MN}(\boldsymbol{x})$ [$\frac{1}{2}d(d+1)$ components in d dimensions], where $\boldsymbol{x}$ is a local coordinate on the given "initial" space-like hypersurface. As we will see in a moment, not all of these variables are dynamical, but four of them are Lagrange multipliers, leading to constraints. These Lagrange multipliers, called "lapse" and "shift" functions (see below), reflect the invariance of the theory under diffeomorphisms in space–time and appear in the equations determining the time evolution of the initial space-like geometry (and therefore also determine which points in space–time are space-like or time-like with respect to each other).

Below, we will use the vierbein $E_M{}^A$ instead of the metric; there is then an extra local symmetry, namely local Lorentz ($= \mathrm{SO}(1,3)$) rotations acting on the flat index A. It is, of course, well known that gravity can be thought of as a gauge theory which is invariant under local translations (i.e. diffeomorphisms), which can be generated by space-like or time-like vector fields. Consequently, we have four canonical constraints, three of which are associated with diffeomorphisms acting on the space-like hypersurface. It is the presence of the constraint associated with the invariance under time-like diffeomorphisms which is responsible for the difference between ordinary gauge theories and gravity: it contains dynamics whereas the other constraints are only "kinematical" (so canonical Yang–Mills theories have only "kinematical" constraints). After quantization, the corresponding constraint becomes the celebrated Wheeler–DeWitt (WDW) equation [12,13]. We will refer to this constraint as the "Hamiltonian constraint", or simply the "WDW operator", regardless of whether we are dealing with the classical or quantum theory. For the remainder of this section we will work in an arbitrary number $(= d)$ of dimensions and only return to four dimensions in the next section.

To proceed, we parametrize the vielbein as follows:

$$E_M{}^A = \begin{pmatrix} N & N_a \\ 0 & e_{ma} \end{pmatrix} \quad \Leftrightarrow \quad E_A{}^M = \begin{pmatrix} N^{-1} & -N^{-1}N^m \\ 0 & e_a{}^m \end{pmatrix}, \tag{2.3}$$

where partial use has been made of the local Lorentz symmetry to eliminate some off-diagonal components. Note that we can write flat space-like indices always as lower indices since they are contracted by δ_{ab}. There is still a residual symmetry under local $SO(d-1)$ rotations of the dreibein e_{ma}. Computing the metric from this parametrization, we find

$$g_{MN} = \begin{pmatrix} N_aN_a - N^2 & N_n \\ N_m & g_{mn} \end{pmatrix}, \tag{2.4}$$

which is the standard parametrization introduced in ref. [9]; the functions N and $N^m = e_a{}^m N_a$ are referred to as lapse and shift functions [9,10]. In the following, we will work with the spin connection ω_{MBC}, which is given by

$$\omega_{MBC} = E_B{}^N \nabla_M E_{CN} = \tfrac{1}{2}\left(\Omega_{ABC} - \Omega_{BCA} + \Omega_{CAB}\right) E_M{}^A, \tag{2.5}$$

where ∇_M is covariant with respect to g_{MN}, so

$$\nabla_M V_N = \partial_M V_N - \Gamma^P{}_{MN} V_P \tag{2.6}$$

for space–time vectors V with the Christoffel connection Γ, and Ω_{ABC} are the so-called coefficients of anholonomy,

$$\Omega_{ABC} = 2E_{[A}{}^M E_{B]}{}^N \partial_M E_{NC}, \tag{2.7}$$

where the brackets indicate antisymmetrization with strength one. The Riemann tensor in our conventions is given by

$$R_{MNAB} = \partial_M \omega_{NAB} - \partial_N \omega_{MAB} + \omega_{MA}{}^C \omega_{NCB} - \omega_{NA}{}^C \omega_{MCB}, \tag{2.8}$$

from which the Ricci tensor and the curvature scalar can be computed by contraction with the inverse vielbein. For the canonical analysis as well as the dimensional reduction of Einstein's theory to lower dimensions it is convenient to express the Einstein action directly in terms of the coefficients of anholonomy. The Einstein action is

$$\mathcal{L} = \tfrac{1}{4} E R = \tfrac{1}{4} E E^{MA} [\nabla_N, \nabla_M] E_A{}^N. \tag{2.9}$$

Making use of the covariant constancy of the vielbein (i.e., $\nabla_M E_N{}^A + \omega_M{}^{AB} E_{NB} = 0$), we can rewrite this action in the form

$$\tfrac{1}{2}E\, \nabla_{[M} E^{MA}\, \nabla_{N]} E_A{}^N = \tfrac{1}{4}E\, \left(\omega_A{}^{AB}\omega^C{}_{CB} - \omega_{ABC}\omega^{BAC}\right) \tag{2.10}$$

up to a total derivative. Inserting (2.5) we get

$$\mathcal{L} = -\tfrac{1}{16}E\left(\Omega_{ABC}\Omega^{ABC} - 2\Omega_{ABC}\Omega^{BCA} - 4\Omega_{AB}{}^B\Omega^{AC}{}_C\right). \tag{2.11}$$

For the canonical analysis we must now write out the coefficients of anholonomy in terms of the parametrization introduced above to find out which of its components contain time derivatives. One verifies that

$$\begin{aligned} \Omega_{abc} &= 2e_{[a}{}^m e_{b]}{}^n \partial_m e_{nc}\,, \\ \Omega_{ab0} &= 0\,, \\ \Omega_{0b0} &= -e_b{}^n N^{-1}\partial_n N\,. \end{aligned} \tag{2.12}$$

Manifestly, these expressions contain no time derivatives. On the other hand,

$$\Omega_{0bc} = N^{-1}\left(e_b{}^n\,(\partial_t - N^m\partial_m)\, e_{nc} - e_b{}^m e_{nc}\partial_m N^n\right) \tag{2.13}$$

does contain time derivatives, but only on the dreibein. Thus, the lapse and shift functions have vanishing momenta and will act as Lagrange multipliers. Substituting (2.12) into (2.11), we get

$$\begin{aligned} \mathcal{L} = \tfrac{1}{16}Ne\big(&4\Omega_{0(bc)}\Omega_{0(bc)} - 4\Omega_{0dd}\Omega_{0ee} - 8\Omega_{a00}\Omega_{acc} \\ &- \Omega_{abc}\Omega_{abc} + 2\Omega_{abc}\Omega_{bca} + 4\Omega_{abb}\Omega_{acc}\big)\,, \end{aligned} \tag{2.14}$$

where (ab) denotes symmetrization in the indices a, b with strength one; after a partial integration in $d-1$ dimensions, the last three terms in parentheses add up to the curvature scalar $R^{(d-1)}$ of the space-like hypersurface plus a term where the derivative acts on N, but this is canceled by the term involving $\Omega_{a00}\Omega_{acc}$. So we get

$$\mathcal{L} = \tfrac{1}{4}Ne\left(\Omega_{0(bc)}\Omega_{0(bc)} - \Omega_{0dd}\Omega_{0ee} + R^{(d-1)}\right). \tag{2.15}$$

The canonical momenta associated with the dreibein are

$$p_a{}^m = \delta\mathcal{L}/\delta\dot{e}_{ma} = \tfrac{1}{2}ee_b{}^m\left(\Omega_{0(ab)} - \delta_{ab}\Omega_{0cc}\right). \tag{2.16}$$

Note the position of indices here: to move indices up or down, or to convert flat into curved indices and vice versa, one contracts with the dreibein. This is important when computing canonical brackets since momentum variables with indices in a position different from the one indicated in (2.16) no longer commute in general. Observe that p transforms as a tensor density with respect to reparametrizations of the space-like hypersurface. Since (2.16) contains only the symmetrized part of Ω_{0ab}, we immediately obtain the primary constraint

$$L_{ab} = e_{m[a}p_{b]}{}^m \approx 0\,, \tag{2.17}$$

which will turn out to be the canonical generator of spatial Lorentz transformations. The remaining momenta vanish:

$$p_0{}^t = \delta\mathcal{L}/\delta\dot{N} = 0, \quad p_a{}^t = \delta\mathcal{L}/\delta\dot{N}_a = 0. \tag{2.18}$$

As explained above, the vanishing of these momentum components implies further constraints, which will be presented in a moment. We can invert the above relations (2.16) to express the components of Ω containing time derivatives by the canonical momenta. In this way we obtain (remember that a, b are spatial indices, so $\delta_{aa} = d - 1$)

$$\Omega_{0(ab)} = \frac{2}{e}\left(p_{ab} + \frac{1}{2-d}\delta_{ab}p\right), \tag{2.19}$$

where $p \equiv e_{ma}p_a{}^m$. This is the extrinsic curvature k_{ab} of the space-like hypersurface, which is defined as the projection of the four-dimensional covariant derivative of the normal vector onto the hypersurface:

$$k_{ab} \equiv E_a{}^M E_b{}^N \nabla_M E_{N0}. \tag{2.20}$$

A short calculation (replacing the ordinary derivative in (2.7) by ∇) shows that this is indeed equal to $\Omega_{0(ab)}$. Taking the trace, we obtain

$$\Omega_{0cc} = \frac{2}{2-d}e^{-1}p. \tag{2.21}$$

The canonical Hamiltonian is thus

$$\mathcal{H} = p_a{}^m \dot{e}_{ma} - \mathcal{L}. \tag{2.22}$$

Use of the above relations and some rearrangement leads to

$$\mathcal{H} = N\mathcal{H}_0 + N_a\mathcal{H}_a, \tag{2.23}$$

where

$$\mathcal{H}_0 \equiv e^{-1}\left(p_{ab}p_{ab} + \frac{1}{2-d}p^2\right) - \tfrac{1}{4}eR^{(d-1)}. \tag{2.24}$$

Furthermore,

$$\mathcal{H}_a \equiv \mathcal{D}_m p_a{}^m, \tag{2.25}$$

where $\mathcal{D}_m$ is covariant with respect to space-like diffeomorphisms, so the dreibein is constant under $\mathcal{D}_m$ and

$$\mathcal{D}_m p_{ab} = e\partial_m(e^{-1}p_{ab}) + \omega_{mac}p_{cb} + \omega_{mbc}p_{ac} \tag{2.26}$$

(there is an extra contribution involving the derivative of the dreibein determinant because p_{ab} is a density). According to the general theory and what has been said above, $\mathcal{H}_0$ and $\mathcal{H}_a$ must vanish weakly in order to preserve $p_0{}^t = p_a{}^t = 0$. These, then, are the canonical constraints, and $\mathcal{H}_0$ is the WDW operator. Consequently, the Hamiltonian vanishes weakly, as is the case in any theory invariant under reparametrizations of the time coordinate (another example is the point particle). For completeness, we mention that the Hamiltonian above is not unique since we may add a term proportional to the Lorentz constraint (2.17) to it with a suitable Lagrange multiplier.

The basic Poisson brackets are given by

$$\{e_{ma}(\boldsymbol{x}), p_b{}^n(\boldsymbol{y})\} = \delta_{ab}\delta_m^n\,\delta^{(2)}(\boldsymbol{x},\boldsymbol{y})\,. \tag{2.27}$$

Other commutators involving the inverse dreibein, or the momenta with the indices in other positions follow in a straightforward fashion. For instance,

$$\{e_{ma}(\boldsymbol{x}), p_{nb}(\boldsymbol{y})\} = e_{na}e_{mb}\,\delta^{(2)}(\boldsymbol{x},\boldsymbol{y})\,. \tag{2.28}$$

This concludes our brief discussion of canonical gravity in the metric formalism (for a much more thorough treatment, the reader is referred to ref. [1]). To quantize the theory, it would now seem that all that remains to be done is to replace the Poisson brackets (2.27) by quantum commutators, or equivalently, the canonical momentum $p_a{}^m(\boldsymbol{x})$ by the functional differential operator $-i\hbar\delta/\delta e_{ma}(\boldsymbol{x})$. However, it is immediately clear that the equations obtained in this way are highly non-linear, and that there will be severe operator ordering ambiguities. While it is still comparatively easy to solve the spatial diffeomorphism constraints by building wave functionals out of invariant integrals of spatial curvatures over the space-like hypersurface, attempts at constructing general solutions of the WDW equation appear completely hopeless. This leaves us with the unappetizing possibility of truncating the WDW equation by only retaining a finite number of degrees of freedom (such as, e.g., the radius of the universe; this is the so-called "mini-superspace" approximation) or of resorting to unilluminating weak or strong coupling limits where either the "kinetic term" involving p^2 or the "potential" $eR^{(d-1)}$ is discarded.

Before turning to the new formulation, one further point should be mentioned. Even within the classical framework, and independently of the phase space variables one chooses, there remains a major unsolved problem in canonical gravity, namely the construction of non-trivial observables in the sense of Dirac. By such an observable we generally mean any functional of the phase space variables that weakly commutes with all the constraints and does not vanish on the constraint hypersurface (otherwise, the constraints themselves would be observables). In other words, we would like to explicitly construct a phase space functional $\mathcal{O}(e_{ma}, p_a{}^m)$ for which

$$\{\mathcal{H}_0(\boldsymbol{x}), \mathcal{O}\} \approx 0\,, \quad \{\mathcal{H}_m(\boldsymbol{x}), \mathcal{O}\} \approx 0\,. \tag{2.29}$$

Unfortunately, for pure gravity, no such functional is known. The main culprit for this sorry state of affairs is again the Hamiltonian constraint; since it contains dynamics, the construction of observables would amount to the identification of "constants of motion" for Einstein's equations. The lack of such observables severely affects the quantum theory, where the notion of observable is indispensable for the physical interpretation of the formalism. As we will see later, the situation is not quite as bad for pure gravity and supergravity in three dimensions, which are topological theories, and matter coupled theories of (super)gravity in higher dimensions which possess "hidden symmetries". In both cases it is possible to construct observables obeying (2.29).

3. Ashtekar's variables

The preceding discussion has clearly led to an impasse, and at this point one might be inclined to believe that some radically new idea is needed to make further progress (such is the attitude of a die-hard string theorist). It was therefore quite a surprise when, in 1986, Ashtekar discovered a *new* set of phase space variables [14] in terms of which not only the non-linear constraints become polynomial, but solutions to *all* the quantized constraints could be found [17]. These solutions are very different from the approximate solutions of the minisuperspace approximation. It is plausible that their unfamiliar form and the fact that they are difficult to interpret simply reflect genuine features of quantum gravity, which we would anyhow expect to be very unusual. In this chapter, we will present a (hopefully) pedestrian introduction to Ashtekar's formalism, although our treatment of the solutions will be rather cursory (these are extensively discussed in recent reviews [2]). An important property (drawback?) of the formalism is that, so far, it only works in space–time dimensions $d = 4$ and $d = 3$ (however, see ref. [18] for some speculations concerning $d > 4$). We will first discuss the case $d = 4$, corresponding to the real (albeit empty) world. The simpler $d = 3$ theory will be dealt with in the following section.

3.1. CONSTRUCTION OF THE NEW VARIABLES

The basic idea leading to the new variables is quite simple. Noticing that the constraint generators (2.24) and (2.25) are schematically of the form "$\partial\omega + \omega^2 + p^2$" and "$(\partial + \omega)p$", we will try to combine them into an expression of the form "$\partial(\omega + p) + (\omega + p)^2$" by introducing a generalized connection field of the form $A =$ "$\omega + p$" (the "Ashtekar connection"), where p is just the momentum variable introduced in (2.16). Let us therefore proceed from the ansatz

$$A_{ma} = -\tfrac{1}{2}\epsilon_{abc}\omega_{mbc} + \alpha\widehat{p}_{ma}\,, \tag{3.1}$$

where

$$\widehat{p}_{ma} \equiv e^{-1}\,(p_{ma} + \beta e_{ma}p)\ . \tag{3.2}$$

The coefficients α and β are to be determined. Defining a new covariant derivative D_m with respect to the Ashtekar connection (3.1) and evaluating it on a spatial Lorentz vector V_a, we get

$$D_m V_a \equiv \partial_m V_a + \epsilon_{abc} A_{mb} V_c = \partial_m V_a + \omega_{mab} V_b + \cdots , \tag{3.3}$$

where the dots indicate terms depending on p. As for the terms containing ω, this agrees with the usual Lorentz covariant derivative (it is for this reason that we have inserted a factor of $-\frac{1}{2}$ in the above definition). Note that the position of the indices on p is not the one of (2.16). Obviously, it is the presence of the epsilon tensor in this expression that forces us to put $d = 4$. The only other possibility is $d = 3$ as we will see, but let us stick with $d = 4$ for the moment. Since we would like A_{ma} to be a proper canonical variable, we demand

$$\{A_{ma}(\boldsymbol{x}), A_{nb}(\boldsymbol{y})\} = 0 . \tag{3.4}$$

Obviously, the terms that contain p and the ones that do not must vanish separately. In appendix A we will show that

$$\{\widehat{p}_{ma}, \widehat{p}_{nb}\} = 0 \tag{3.5}$$

leads to $\beta = -\frac{1}{2}$. We will also show that there is a functional $G[e_{ma}]$ such that

$$A_{ma} = \{\widehat{p}_{ma}, G\} + \alpha \widehat{p}_{ma} . \tag{3.6}$$

Now it is easy to see that the A's commute: inserting the last equation into (3.4) and making use of (3.5) and the fact that the first term in the commutator depends only on e_{ma}, we get

$$\begin{aligned} \{A_{ma}, A_{nb}\} &= \alpha \{\{\widehat{p}_{ma}, G\}, \widehat{p}_{nb}\} - \alpha \{\{\widehat{p}_{nb}, G\}, \widehat{p}_{ma}\} \\ &= \alpha \{\{\widehat{p}_{ma}, \widehat{p}_{nb}\}, G\} = 0 , \end{aligned} \tag{3.7}$$

where, in the last step, the Jacobi identity has been invoked.

Since the requirement (3.4) does not fix the coefficient α, let us now analyze the constraints in terms of the new variables (of course, we now assume that they can indeed be reexpressed in this way!). We will first work out the field strength of Ashtekar's connection in terms of the "old" variables. This yields

$$\begin{aligned} F_{mna} &\equiv \partial_m A_{na} - \partial_n A_{ma} + \epsilon_{abc} A_{mb} A_{nc} \\ &= -\tfrac{1}{2} \epsilon_{abc} R_{mnbc} + \alpha \left(\mathcal{D}_m \widehat{p}_{na} - \mathcal{D}_n \widehat{p}_{ma} \right) + \alpha^2 \epsilon_{abc} \widehat{p}_{mb} \widehat{p}_{nc} . \end{aligned} \tag{3.8}$$

Contracting once with the inverse dreibein, we get

$$e_a{}^m F_{mna} = \alpha e^{-1} \mathcal{D}_m p_n{}^m , \tag{3.9}$$

where the Bianchi identity $R_{m[abc]} = 0$ and the Lorentz constraint have been used. This is indeed proportional to the the diffeomorphism constraint. The other possibility is

$$\epsilon_{abc} e_a{}^m e_b{}^n F_{mnc} = -R^{(3)} - \alpha^2 e^{-2} \left(p_{ab} p_{ab} - \tfrac{1}{2} p^2 \right) , \tag{3.10}$$

which reduces to the WDW constraint upon multiplication by e and for the values $\alpha = \pm 2\mathrm{i}$. Obviously, there are two possible choices for α, and hence two possible choices for the Ashtekar connection, which we label $A^{(\pm)}$. Note that $A^{(+)}$ and $A^{(-)}$ do *not* commute. Observe also that α is imaginary, so Ashtekar's connection is *complex*. This is certainly an unusual feature as it is like choosing q and $z = q + ip$ as canonically conjugate variables in ordinary mechanics! We must thus supplement the formalism by a reality constraint corresponding to $z + \bar{z} = 2q$ in order to ensure that we end up with the correct number of degrees of freedom. This extra constraint is a somewhat unappealing feature of the formalism. There has been some confusion in the early literature whether this could be a real problem, especially since the reality constraint in some examples seems to be non-polynomial [21]. The consensus at present is that the problem can be consistently dealt with by first analyzing the (complex) equations and then imposing the reality conditions. It is anyhow clear that, at least at the classical level, there should be no problem since the theory is still equivalent to Einstein's theory, which is known to be perfectly consistent. At the quantum level, however, the problem is tied up with some as yet unresolved issues related to the construction of a scalar product in the Hilbert space of quantum states.

There only remains the Lorentz constraint. Anticipating the final result, we evaluate the fully covariant derivative with respect to A on $ee_a{}^m$ to get

$$D_m(ee_a{}^m) = \mathcal{D}_m\,(ee_a{}^m) + e\epsilon_{abc}\widehat{p}_{mb}e_c{}^m\,. \tag{3.11}$$

The first term vanishes by the covariant constancy of the dreibein (this is not entirely trivial because the derivative D_m acts only on the flat index a; the full covariantization is achieved by including the factor e). The second term is nothing but the Lorentz constraint (2.17).

Finally, we have to identify the variable which is canonically conjugate to A. It is just the "densitized" inverse dreibein dreibein $ee_a{}^m$ introduced above. A short calculation confirms that

$$\{\tilde{e}_a^m(\boldsymbol{x}), \widehat{p}_{nb}(\boldsymbol{y})\} = -\delta_{ab}\delta_n^m\,\delta^{(2)}(\boldsymbol{x},\boldsymbol{y}) \tag{3.12}$$

and therefore

$$\{\tilde{e}_a^m(\boldsymbol{x}), A_{nb}(\boldsymbol{y})\} = -2\mathrm{i}\delta_{ab}\delta_n^m\,\delta^{(2)}(\boldsymbol{x},\boldsymbol{y})\,, \tag{3.13}$$

where we have defined $\tilde{e}_a^m \equiv ee_a{}^m$. To summarize, we have now succeeded in reformulating the whole theory in terms of the new connection $A_m{}^a$ and its canonically conjugate variable $\tilde{e}_a^m$. The constraints are simply given by

$$\begin{aligned} \tilde{e}_a^m F_{mn}{}^a &\approx 0\,, \\ \epsilon^{abc}\tilde{e}_a^m\tilde{e}_b^n F_{mnc} &\approx 0\,, \\ D_m\tilde{e}_a^m &= 0\,, \end{aligned} \tag{3.14}$$

and are thus manifestly polynomial unlike the original expressions (2.25) and (2.24). This result constitutes a major simplification and facilitates the search for solutions to the quantum constraints. An extra advantage is that we do not need to impose the requirement that the new variable $\tilde{e}_a^m$ be invertible in contrast to the usual metric formalism, where both the metric and its inverse are required for the formulation of the canonical constraints. This is in accord with speculations [22] that prior to the emergence of a "semi-classical" space–time characterized by a reference background metric, there should be a "topological phase" of quantum gravity characterized by a singular vacuum expectation value of the metric (or the vielbein). We also note that the above expressions provide a realization of the canonical constraints of gravity on the phase space of an SO(3) Yang–Mills theory (see, e.g., ref. [23] for a further discussion of this point).

The original definition of the new variables [14,15] is actually slightly different from the one employed above, although, of course, equivalent. To recover the variables as defined there, we contract the densitized inverse dreibein and the connection field A with Pauli matrices according to

$$\begin{aligned} \tilde{e}^m_{\alpha\beta} &\equiv \tilde{e}^m_a \sigma_{a\alpha\beta} \,, \\ A_{m\alpha\beta} &\equiv A_{ma}\sigma_{a\alpha\beta} \,, \end{aligned} \tag{3.15}$$

replacing Lorentz indices by spinorial SU(2) indices. The field $\tilde{e}^m_{\alpha\beta}$ is referred to as a "soldering form" (it "solders" upper world indices to spinorial tangent space indices). In terms of these variables, the Poisson brackets become

$$\{\tilde{e}^m_{\alpha\beta}(\boldsymbol{x}), A_{n\gamma\delta}(\boldsymbol{y})\} = \Big(2\delta_{\alpha\delta}\delta_{\gamma\beta} - \delta_{\alpha\beta}\delta_{\gamma\delta}\Big)\delta^m_n\,\delta^{(2)}(\boldsymbol{x},\boldsymbol{y})\,. \tag{3.16}$$

The reason for this slightly different choice is that Ashtekar's connection can be understood as originating from the *four*-dimensional spin connection: the connection (3.15) is nothing but the "pullback" of the $d = 4$ spin connection to the space-like hypersurface (this observation is also useful as a mnemonic device). To see this, we contract the spin connection with a γ matrix and write the result according to

$$\omega_{mAB}\gamma^{AB} = \omega_{mab}\gamma_{ab} + 2\omega_{ma0}\gamma_a\gamma_0\,. \tag{3.17}$$

Using the formula $\omega_{ab0} = \Omega_{0(ab)}$ and

$$\gamma_{ab} = -\mathrm{i}\epsilon_{abc}\begin{pmatrix}\sigma_c & 0\\ 0 & \sigma_c\end{pmatrix}, \quad \gamma_0\gamma_a = \begin{pmatrix}\sigma^a & 0\\ 0 & \sigma^a\end{pmatrix}, \tag{3.18}$$

we just get Ashtekar's connection. The freedom in choosing the sign of the coefficient α precisely corresponds to the two chiralities of the spinor on which the connection field acts.

The fact that two connections A_{ma} have vanishing Poisson brackets, whose direct proof was not entirely obvious, can now be regarded as a consequence of this observation and the fact that the Lorentz covariant derivative appearing in supergravity theories coincides with (3.3). As is well known (and as we will see in sections 4 and 5), the invariance of the supergravity action under local supersymmetry transformations implies the existence of a further constraint, with the time component of the gravitino acting as the Lagrange multiplier. This constraint contains the combination $D_{[m}\psi_{n]}$, where D_m is the pullback of the four-dimensional spin connection appearing in (3.17) and thus precisely the covariant derivative with respect to Ashtekar's connection. Since two local supersymmetry transformations commute to give a translation, the canonical bracket between two supersymmetry constraints should give rise to (amongst other things) the diffeomorphism and the Hamiltonian constraints. In the actual computation, these constraint operators arise from commuting two fermions and must therefore be expressible in terms of the field strength (3.8). In other words, consistency of the canonical formalism with local supersymmetry automatically implies the results derived above!

3.2. SOLUTION OF QUANTUM CONSTRAINTS

As is well known, the quantization procedure consists in replacing the classical Poisson (or Dirac) brackets by quantum commutators, or, equivalently, in replacing the momentum variables by differential operators. There is, however, some ambiguity here since it is by no means clear for a highly non-linear theory such as Einstein's theory which variables one should replace in this way. For instance, in the metric formalism discussed in section 2, the most natural choice is to replace $p_a{}^m$ by $-i\hbar\delta/\delta e_{ma}$, as we have already discussed, but one could equivalently choose to work with the inverse dreibein and its canonically conjugate variable instead. It is also clear that different choices may be expected to lead to inequivalent quantum theories, a phenomenon that is already known from flat space quantum field theories [24]. In fact, it is precisely the hope that quantization in terms of the new variables may lead to a theory that is somehow "better" defined than quantum gravity in the metric formalism, which leads to basically intractable equations. In this section, we will briefly discuss the construction of solutions to the Hamiltonian constraint. The first solutions of this type were obtained in ref. [16]; unfortunately, they are not annihilated by the diffeomorphism generator. Nonetheless, that work constituted considerable progress, since it was the WDW constraint which had resisted all previous attempts at solution, not the diffeomorphism constraint. In subsequent work [17], a more abstract framework was introduced, where from the outset one deals with diffeomorphism invariant objects (knot and link classes). This requires the consideration of yet new and more exotic phase space quantities, the "loop variables", and the reformulation of the constraints and the canonical brackets in terms of them.

In accordance with ref. [16], we take Ashtekar's connection as the basic variable and replace

$$\tilde{e}_a^m(\boldsymbol{x}) \longrightarrow -\delta/\delta A_{ma}(\boldsymbol{x}) \tag{3.19}$$

(we set $\hbar = 1$). An unusual feature is that the inverse densitized metric is now represented by a differential operator, i.e.,

$$gg^{mn}(\boldsymbol{x}) = \frac{\delta}{\delta A_{ma}(\boldsymbol{x})}\frac{\delta}{\delta A_{na}(\boldsymbol{x})}. \tag{3.20}$$

To obtain the metric itself, one would thus have to solve this relation for g_{mn}—clearly not an easy task! In addition, there may be a problem with short distance singularities resulting from the clash of two functional differential operators at coincident points, which will show up in a factor $\delta^{(2)}(0)$. We will ignore this difficulty for the moment, but it is plainly evident that whatever solutions to the quantum constraints can be found, they will not be easy to interpret.

In making the transition to the quantum theory, we must also decide how to order the operators. For instance, it will now matter whether the differential operators in the Hamiltonian constraint are placed to the left or to the right of $F_{mna}(A)$. The first possibility was considered in ref. [15], whereas the second one underlies the work of ref. [16]. We will adopt the second prescription and put the operators *to the right* for the moment. With this choice of ordering, the WDW constraint becomes

$$\epsilon_{abc}\tilde{e}_a^m\tilde{e}_b^n F_{mnc} \longrightarrow \epsilon^{abc}F_{mnc}(A(\boldsymbol{x}))\frac{\delta}{\delta A_{ma}(\boldsymbol{x})}\frac{\delta}{\delta A_{nb}(\boldsymbol{x})}. \tag{3.21}$$

For the construction of solutions it proves convenient to first consider the Lorentz constraint. As is well known, the basic Lorentz invariant wave functionals are the Wilson loops. We parametrize a closed loop γ by a function $\gamma^m(s)$; the parameter s is normalized by requiring $\gamma(s) = \gamma(s+1)$ (we will assume the function $\gamma(s)$ to be periodic in s, so that the base point of the loop can be freely shifted). The Wilson loop is

$$\Psi_\gamma[A] = \operatorname{Tr}\mathcal{P}\exp\oint_\gamma A, \tag{3.22}$$

where $\mathcal{P}$ denotes path ordering from right to left along the orientation of the loop (the dependence on the base point of the loop drops out in the trace). To evaluate the functional derivatives on this expression, we need a little more notation; we define

$$U_\gamma(s',s) \equiv \mathcal{P}\exp\int_s^{s'} \mathrm{d}t\,\dot{\gamma}^m(t)A_m(\gamma(t)), \tag{3.23}$$

so that $\Psi_\gamma[A] = \operatorname{Tr}U_\gamma(1,0)$ ·[for the sake of clarity, we will sometimes write (3.23) as $U(\gamma(s'),\gamma(s))$]. $U_\gamma(s+1,s)$ is therefore the holonomy at the point $\gamma(s)$ defined by parallel transporting the connection $A = \mathrm{d}x^m A_{ma}\sigma^a$ from $\gamma(s)$ to $\gamma(s+1)$, i.e. once around the loop γ.

Acting twice on the Wilson loop functional (3.22) with the operator (3.19), we obtain

$$\frac{\delta}{\delta A_{ma}(\boldsymbol{x})}\frac{\delta}{\delta A_{nb}(\boldsymbol{y})}\Psi_\gamma[A] \tag{3.24}$$

$$\begin{aligned}
&= \int_0^1 \mathrm{d}s\,\dot{\gamma}^m(s)\int_0^s \mathrm{d}s'\,\dot{\gamma}^n(s')\delta^{(3)}(\boldsymbol{x},\gamma(s))\delta^{(3)}(\boldsymbol{y},\gamma(s')) \\
&\quad\times \mathrm{Tr}\Big(U_\gamma(1,s')\sigma_a U_\gamma(s',s)\sigma_b U_\gamma(s,0)\Big) \\
&\quad+ \int_0^1 \mathrm{d}s\,\dot{\gamma}^n(s)\int_0^s \mathrm{d}s'\,\dot{\gamma}^m(s')\delta^{(3)}(\boldsymbol{y},\gamma(s))\delta^{(3)}(\boldsymbol{x},\gamma(s')) \\
&\quad\times \mathrm{Tr}\Big(U_\gamma(1,s')\sigma_b U_\gamma(s',s)\sigma_a U_\gamma(s,0)\Big).
\end{aligned} \tag{3.25}$$

When $\boldsymbol{x}$ and $\boldsymbol{y}$ coincide, this expression becomes *symmetric* in the indices a and b, and m and n, respectively. To take care of the divergent factor $\delta^{(2)}(0)$ which arises in this limit, one regulates the δ function, for instance by fattening the loop into a sausage [16] [#2]. We now see that the above expression vanishes upon contraction with the factor $\epsilon_{abc}F_{mnc}$ by antisymmetry. It is easy to see that this mechanism does not work for the diffeomorphism generators, for which the field strength is not contracted with an antisymmetric tensor, as is evident from (3.14). The Wilson loop is thus annihilated by the Lorentz constraint and the WDW constraint, but not invariant under diffeomorphisms. It may seem paradoxical that a solution of the WDW constraint does not also solve the diffeomorphism constraints, since it is known that—at least at the level of Poisson brackets—the commutator of two suitably weighted WDW operators should produce amongst other things a spatial diffeomorphism. However, detailed analysis shows that this is not quite true because of ordering subtleties [16]: the structure functions (*not* constants!) on the right hand side of the commutator appear to the *right* of a differential operator, and therefore give rise to extra unwanted contributions. This is a direct consequence of our choice of operator ordering [#3].

[#2] Evidently, this regularization must preserve the symmetry under interchange of a and b, or m and n, since otherwise any result can be obtained. This is a weak point in the argument.

[#3] Experience with string theory teaches us that we should even anticipate difficulties of this kind. The Virasoro algebra, which is nothing but the algebra of the canonical constraints associated with reparametrization invariance on the two-dimensional world sheet, contains an anomalous central term that is rooted in very similar ordering ambiguities. The central term is, in fact, responsible for much of the non-trivial structure of string theory, as is well known [7]. Moreover, because of the central term, it is inconsistent to impose the full set of Virasoro constraints on the physical states; rather one uses only half of them in a Gupta–Bleuler formulation. There is consequently no reason to expect that the ordering problems of canonical gravity in four dimensions can be easily resolved; on the contrary, one would

To overcome the difficulties with the diffeomorphism constraint and to construct solutions to *all* the constraints, one might now try, for instance, to "average" the wave functional $\Psi_\gamma[A]$ over all loops which are diffeomorphic to γ. This is, however, a difficult task to perform in practice, since no suitable (and manageable) measure in the infinite-dimensional space of diffeomorphisms is known. A better way out, proposed in ref. [17], is to switch from the above "connection representation" to the so-called "loop representation", where the basic objects are no longer functionals of the Ashtekar connection and their canonically conjugate variables, but of loops (or knots and links). Although there is no room to discuss this approach in detail here, we sketch the basic idea. To begin with, one considers new variables in phase space in addition to the Wilson loop, namely loops with multiple insertions of the canonical variable $\tilde{e}_a^m$ along the loop. To systematize the notation, we denote the basic Wilson loop by $T^0[\gamma]$, and define generalized loop variables T^n with n insertions of $\tilde{e}_a^m$ at the points $\gamma(s_1), ..., \gamma(s_n)$ on the loop (which we assume to be "time ordered" from right to left along the orientation of the curve) by

$$T^n[\gamma; s_n, ..., s_1]^{m_n \cdots m_1} \equiv \operatorname{Tr} U_\gamma(1, s_n)\tilde{e}_{a_n}^{m_n}(\gamma(s_n))\sigma^{a_n} U_\gamma(s_n, s_{n-1}) \cdots \\ \times \cdots U_\gamma(s_2, s_1)\tilde{e}_{a_1}^{m_1}(\gamma(s_1))\sigma^{a_1} U(s_1, 0) . \quad (3.26)$$

An important result is that, with respect to the Poisson brackets, these loop variables form a *closed algebra*. If two loops do not intersect, the Poisson bracket of the corresponding loop variables clearly vanishes; otherwise, the result is a linear combination of loop variables which are based on new loops formed by joining the loops at the points of intersection (see ref. [17] for a detailed description of the rules). The strategy is now to "forget" how these variables were derived and how their Poisson brackets were computed, and to take the loop variables and their associated Poisson algebra as fundamental. In this way one arrives at a formulation which no longer makes reference to the original phase space variables (this transition to a new representation can be thought of as some kind of Fourier transformation).

An advantage of this proposal is that one is working with diffeomorphism invariant objects (knots and links) from the outset. On the other hand, it is now much more difficult to obtain explicit representations of the quantities one is dealing with, since all operations must be defined in terms of loop variables. For instance, the WDW operator is no longer given by an explicit expression like (3.14), but rather as a kind of area derivative of the loop [17]. Quantization now also works in a different way. Rather than to replace the original phase space variables $\tilde{e}_a^m$ and $A_m{}^a$ by differential operators, one "quantizes" the loop

expect the requirement of closure to give rise to severe constraints on the allowed theories, as is the case in string theory. This point has also been emphasized by F. Englert (private communication).

algebra directly by replacing the Poisson brackets of T^n by quantum commutators. The resulting quantum theory is quite different from quantum gravity in the connection representation. Since invariance under diffeomorphisms and local Lorentz transformations have been built in from the start, it only remains to verify that the WDW constraint is satisfied. This requires a bit of technical trickery which we will not go into here, however.

Before closing this section, we would like to mention a potentially serious drawback of all these solutions, which was first pointed out in ref. [25]. As it turns out, all of them are annihilated by the operator representing the dreibein determinant! This is most easily seen in the connection representation, where the metric determinant is represented by the operator

$$g(\boldsymbol{x}) = \epsilon_{abc}\epsilon_{mnp}\frac{\delta}{\delta A_{ma}(\boldsymbol{x})}\frac{\delta}{\delta A_{nb}(\boldsymbol{x})}\frac{\delta}{\delta A_{pc}(\boldsymbol{x})}. \tag{3.27}$$

This operator annihilates the Wilson loop for the very same reason that ensured the vanishing of the WDW operator on it, namely the fact that one is contracting a symmetric tensor with an antisymmetric one (the problem with the factor $\delta^{(3)}(0)$ is argued away as before). Furthermore, the difficulty cannot be circumvented by allowing for solutions involving linear combinations of an arbitrary number of kinks and self-intersections of the loops as well as (finite) linear superpositions of multiple loops [25]. Although one might think that the degeneracy of the metric is not really such a serious problem because the metric (and hence its determinant) is not an observable, this means in particular that the above expressions solve the constraints regardless of the value of the cosmological constant and thus cannot distinguish between physically very different situations! It is therefore somewhat doubtful whether these solutions are the appropriate ones for the description of conventional physics.

These unsatisfactory features and the desire to find non-degenerate solutions have motivated recent work which is based on the opposite operator ordering prescription (i.e., the differential operators in (3.21) now appear *to the left* of the field strength F). Namely, it can be shown that the exponential of the Chern–Simons action muliplied by the inverse of the cosmological constant solves the WDW constraint with a cosmological constant [26] [the WDW operator is then just the sum of the original expression (3.14) and a term $\Lambda g(\boldsymbol{x})$, which is represented by the operator (3.27)]. This solution is very different from the ones considered above as it is supported on all of the space-like hypersurface, and not just on closed loops. Moreover, it is clearly non-degenerate because it is not annihilated by the operator $\Lambda g(\boldsymbol{x})$. It is then also possible to obtain non-degenerate solutions in the loop represenation. It can be shown that the loop transform, i.e. integration over all connections for a given closed loop γ (with the exponential of the Chern–Simons action playing the role of the factor e^{ikx} for ordinary Fourier transformations) leads to knot and link invariants [27].

Although this result is not completely unexpected in view of the results of ref. [28], it does point to an intriguing and beautiful connection between quantum gravity and knot theory. A further discussion of these issues is unfortunately is beyond the scope of these lectures.

4. Canonical gravity and supergravity in three dimensions

Instead of further dwelling on the four-dimensional theory, we will now take a step back and consider gravity and supergravity in three space–time dimensions (for a general introduction to supergravity and many references, see ref. [6]) #4. The main reason is that the three-dimensional theories provide a setting for the formalism which from our point of view is more natural in several respects. Most importantly, pure gravity and supergravity are topological theories in three dimensions, which means that they do not possess propagating degrees of freedom [22]. Bona fide solutions to all constraints can be found. In addition, genuine observables in the sense of Dirac can be constructed: they are just given by the three-dimensional analogs of the loop variables introduced in the foregoing section and their supersymmetric generalizations. As a consequence, the solutions to the quantum constraints can be directly obtained by applying the observables to a suitable "vacuum" functional, which is just **1** in the bosonic case, and given by formula (4.41) below in the case of supergravity. A technical advantage is that there is no need for a reality constraint in three dimensions.

Three-dimensional gravity and supergravity have already been studied in the past. The "physics" of pure gravity (absence of gravitational excitations in empty space, conical singularities at the locations of mass sources, etc.) were first examined in refs. [30–32]. Since Einstein's action is superficially non-renormalizable in three dimensions, the theory was for a long time thought to make no more sense as a quantum theory than gravity in four dimensions. The fact that pure gravity in three dimensions is a topological theory with a *finite*-dimensional phase space and hence can be solved completely came thus as quite a surprise [22]. An essential ingredient in that construction was the reformulation of Einstein's theory as a Chern–Simons gauge theory. This new version of the theory and the Wilson loop observables were further studied in refs. [33–35]. In this is section, we will, however, not make use of this formulation, but rather adopt the version of ref. [36], which is a direct extension of Ashtekar's formalism to pure gravity in three dimensions, and hence closer to the main subject of these lectures. Needless to say that the "physics" is the same in both formulations. As for supergravity, our results are also not entirely new, with the exception of

#4 The first canonical treatment of supergravity in four dimensions was given in ref. [29]. In the context of Ashtekar's new variables, it was first discussed in ref. [21].

the solutions to the quantum supergravity constraints presented in section 4.2. Wilson loop observables in supergravity were already studied in ref. [37]. However, that work uses a superspace formulation, so the Wilson loop is defined as a supertrace, whereas it is explicitly written out in components here. Furthermore, our observation that the fermionic topological modes are related to (the fermionic part of) super-Teichmüller space appears to be new.

As for notation, we will switch gears slightly by now using Greek letters $\mu, \nu, \ldots$ for curved indices in the three-dimensional space–time of signature $(-++)$, but will continue to use letters $a, b, \ldots$ for the tangent space indices transforming under SO(1,2), which now of course will appear as upper and lower indices. We will use the Levi-Civita tensor with $\epsilon^{012} = -\epsilon_{012} = +1$. Also, we will use letters $i, j, \ldots$ to denote space-like (now two-dimensional) curved indices, and letters $\alpha, \beta, \ldots = 1, 2$ to denote spinor indices transforming under $\mathrm{SL}(2,\mathbb{R}) \cong \mathrm{SO}(1,2)$.

4.1. LAGRANGIAN, CANONICAL VARIABLES AND CONSTRAINTS

One of the main benefits of the discussion in the the preceding section is that we do not have to repeat the laborious derivation of Ashtekar's connection from the metric formalism given there. Rather we will now make a shortcut by exploiting the fact that this connection coincides with the pullback of the spin connection to the (now two-dimensional) space-like hypersurface Σ, making the usual assumption that the three-dimensional space–time is topologically equivalent to $\mathbb{R} \times \Sigma_g$, where Σ_g a is two-dimensional manifold (Riemann surface) with g handles. It is convenient to use the (first order) dualized spin connection

$$A_\mu{}^a = -\tfrac{1}{2}\epsilon^{abc}\omega_{\mu bc}\,, \tag{4.1}$$

in terms of which the Lorentz [i.e. SO(1,2)] covariant derivative acting on a two-component spinor ϵ reads

$$D_\mu\epsilon = \left(\partial_\mu + \tfrac{1}{2}\gamma_a A_\mu^a\right)\epsilon\,. \tag{4.2}$$

The other relevant variable is, of course, the dreibein $e_\mu{}^a$. Observe that, in contradistinction to the four-dimensional case discussed before, we do not commit ourselves to a special Lorentz gauge for the dreibein here; this means that instead of a subgroup of the full Lorentz group we retain the full Lorentz group in three dimensions, which is SO(1,2). The fact that we are dealing with SO(1,2) rather than SO(3) is the principal reason that we do not have to impose a reality constraint in this case because the spinor represention SO(1,2) [which is nothing but $\mathrm{SL}(2,\mathbb{R})$] is real. One can use the γ matrices $\gamma_0 = i\sigma_2$, $\gamma_1 = \sigma_1$ and $\gamma_2 = \sigma_3$, which leads to

$$\gamma_a\gamma_b = \eta_{ab}\mathbf{1} - \epsilon_{abc}\gamma^c\,. \tag{4.3}$$

For the field strength, we have the same formula as before, but with a different sign for the quadratic part:

$$F_{\mu\nu a} = -\tfrac{1}{2}\epsilon_{abc}R_{\mu\nu}{}^{bc} = \partial_\mu A_{\nu a} - \partial_\nu A_{\mu a} - \epsilon_{abc}A_\mu{}^b A_\nu{}^c . \tag{4.4}$$

Using the three-dimensional Levi-Civita tensor density, we can now write down the gravitational action

$$\mathcal{L} = \tfrac{1}{4}e\,R = \tfrac{1}{4}\epsilon^{\mu\nu\rho}e_\mu{}^a F_{\nu\rho a} . \tag{4.5}$$

Varying this action with respect to the dreibein, we see that the field strength $F_{\mu\nu a}$ must vanish, in contrast to the four-dimensional situation, where the field strength is non-zero in general. Hence, the connection $A_\mu{}^a$ is pure gauge, at least locally. This shows explicitly that gravity in three dimensions has no propagating degrees of freedom. However, there may be topological degrees of freedom because, globally, the solutions can be non-trivial in the sense that there does not exist a globally defined function g such that $A_\mu = g^{-1}\partial_\mu g$.

The $N = 1$ supergravity Lagrangian is a simple extension of (4.5), since we only need to add a Rarita–Schwinger-type action to it. The Lagrangian reads

$$\mathcal{L} = \epsilon^{\mu\nu\rho}\left(\tfrac{1}{4}e_\mu{}^a F_{\nu\rho a} + \tfrac{1}{2}\overline{\psi}_\mu D_\nu \psi_\rho\right) . \tag{4.6}$$

Here ψ_μ is a two-component Majorana spinor, i.e., $\overline{\psi} = \psi^{\mathrm{T}}C$, where C is the charge conjugation matrix. The covariant derivative D_μ has been defined in (4.2). In addition to being invariant under general coordinate and local Lorentz transformations, the above Lagrangian is invariant under the local supersymmetry transformations

$$\delta\psi_\mu = D_\mu\epsilon, \quad \delta e_\mu{}^a = \bar{\epsilon}\gamma^a\psi_\mu; \quad \delta A_\mu{}^a = 0 . \tag{4.7}$$

As the Rarita–Schwinger term is independent of the dreibein, the previous equation of motion $F_{\mu\nu a} = 0$ for the dreibein remains valid, and therefore the field A is still pure gauge locally. The Rarita–Schwinger equation $\epsilon^{\mu\nu\rho}D_\nu\psi_\rho = 0$ implies that the gravitino field, too, is locally pure gauge, so that the theory describes only topological degrees of freedom. Thus we can always find a locally defined spinor ϕ such that $\psi_\mu = D_\mu\phi$. An obstruction only arises if the spinor ϕ cannot be globally defined. In this case ψ_μ depends on finitely many "supermoduli", so the theory still lives on a finite-dimensional phase space. Variation with respect to the connection $A_\mu{}^a$ tells us that the covariant derivative of the dreibein is equal to a fermionic bilinear (torsion); this equation can be solved for $A_\mu{}^a$ in terms of the dreibein and the fermionic torsion ("second order formalism").

To derive the Hamiltonian, we write the space and time components of (4.6) separately:

$$\mathcal{L} = \epsilon^{ij}\left(\tfrac{1}{4}e_t{}^a F_{ija} - \tfrac{1}{2}e_i{}^a F_{tja} + \overline{\psi}_t D_i\psi_j - \tfrac{1}{2}\overline{\psi}_i D_t\psi_j\right) . \tag{4.8}$$

The canonical momenta associated with the bosonic fields read (here $i, j, \ldots$ denote two-dimensional vector indices)

$$p_a{}^t = \delta\mathcal{L}/\delta\dot{e}_t{}^a = 0, \qquad \Pi_a{}^t = \delta\mathcal{L}/\delta\dot{A}_t{}^a = 0, \tag{4.9}$$

$$p_a{}^i = \delta\mathcal{L}/\delta\dot{e}_i{}^a = 0, \qquad \Pi_a{}^i = \delta\mathcal{L}/\delta\dot{A}_i{}^a = \tfrac{1}{2}\epsilon^{ij}e_{ja}, \tag{4.10}$$

and the fermionic momenta are ($\alpha, \beta, \ldots = 1, 2$ are two-dimensional spinor indices; we use the convention that spinor derivatives always act from the left)

$$\bar{\pi}^t{}_\alpha = \delta\mathcal{L}/\delta\dot{\psi}_{t\alpha} = 0, \tag{4.11}$$

$$\bar{\pi}^i{}_\alpha = \delta\mathcal{L}/\delta\dot{\psi}_{i\alpha} = -\tfrac{1}{2}\epsilon^{ij}\overline{\psi}_{j\alpha}. \tag{4.12}$$

Observe that $\Pi_a{}^i$ and $p_a{}^i$ are rectangular (3 by 2) matrices. The constraints (4.10) and (4.12) are second class. So we have to replace the Poisson brackets by Dirac brackets and then use these constraints to eliminate, e.g., $p_a{}^i$, $\Pi_a{}^i$ and $\bar{\pi}^i$. Doing this, we get the Dirac brackets (labeled by an asterisk) for the remaining canonical variables,

$$\{\psi_{i\alpha}(\boldsymbol{x}), \psi_{j\beta}(\boldsymbol{y})\}_* = \epsilon_{ij}C^{-1}_{\alpha\beta}\delta^{(2)}(\boldsymbol{x},\boldsymbol{y}), \tag{4.13}$$

$$\{e_i{}^a(\boldsymbol{x}), A_j{}^b(\boldsymbol{y})\}_* = 2\epsilon_{ij}\eta^{ab}\delta^{(2)}(\boldsymbol{x},\boldsymbol{y}). \tag{4.14}$$

Observe that ϵ_{ij} is a density of weight -1 and the δ-function is one of weight $+1$. The secondary constraints induced by (4.9) and (4.11) are

$$\mathcal{H}_a \equiv \delta\mathcal{L}/\delta e_t{}^a = \tfrac{1}{4}\epsilon^{ij}F_{ija}, \tag{4.15}$$

$$L_a \equiv \delta\mathcal{L}/\delta A_t{}^a = \epsilon^{ij}\left(D_i e_{ja} - \tfrac{1}{2}\overline{\psi}_i\gamma_a\psi_j\right), \tag{4.16}$$

$$\mathcal{S}_\alpha \equiv \delta\mathcal{L}/\delta\overline{\psi}_{t\alpha} = \epsilon^{ij}D_i\psi_{j\alpha}, \tag{4.17}$$

which are now first class with reference to the Dirac brackets. The Hamiltonian is

$$H = -e_t{}^a\mathcal{H}_a - A_t{}^a L_a - \overline{\psi}_t\mathcal{S}. \tag{4.18}$$

The generator of spatial diffeomorphisms has a very similar form,

$$e_i{}^a\mathcal{H}_a + A_i{}^a L_a + \overline{\psi}_i\mathcal{S}. \tag{4.19}$$

It is also elementary to verify from the canonical brackets given above that the variations of the fields under the respective local symmetries are indeed canonically generated by the constraints (4.15)–(4.17), possibly up to terms proportional to the constraints; for instance, with $\mathcal{S}[\epsilon] = \int\bar{\epsilon}\mathcal{S}$,

$$\delta_\epsilon\psi_i(x) = \{\mathcal{S}[\epsilon], \psi_i(x)\}_*, \tag{4.20}$$

and so on.

4.2. OBSERVABLES AND QUANTUM STATES

Before considering the quantum theory, we examine the classical observables. According to the definition given in section 2, such observables must weakly commute with the canonical constraints. From the discussion at the end of the preceding section, we know that the brackets of the constraints with any functional of the phase space variables just give the variations of this functional under the corresponding local symmetries, possibly up to terms vanishing on the constraint hypersurface. Hence, the observables must be invariant functionals with respect to spatial diffeomorphisms, local Lorentz rotations and supersymmetry, and this is the definition which we will employ in the sequel. To streamline the subsequent discussion somewhat, we will use differential forms in the remainder,

$$e^a = \mathrm{d}x^i\, e_i{}^a\,, \quad A^a = \mathrm{d}x^i\, A_i{}^a\,, \quad \psi = \mathrm{d}x^i\, \psi_i\,, \tag{4.21}$$

which represent sections of vector or spinor bundles over the two-dimensional space-like hypersurface. We find it convenient to define $A \equiv \frac{1}{2}\gamma_a A^a$, which under local Lorentz rotations transforms as $A(x) \to h^{-1}(x)(\mathrm{d} + A(x))h(x)$ with a globally defined $h(x) \in \mathrm{SL}(2,\mathbb{R})$ (the contracted dreibein $\frac{1}{2}\gamma_a e^a$ transforms in exactly the same way). Since the constraint (4.15) implies flatness of the connection A, we conclude that $A = g^{-1}\,\mathrm{d}g$ locally for some matrix function $g(x)$ taking values in $\mathrm{SL}(2,\mathbb{R})$ [under gauge transformations we have $g(x) \to g(x)h(x)$]. The non-triviality of the flat connection A is "measured" (and, in fact, completely characterized) by its holonomies around non-contractible closed loops γ. If the holonomy is non-trivial, i.e. different from unity, $g(x)$ cannot be extended to a globally single-valued function: if we start at some point $\gamma(s)$ with a matrix $g(\gamma(s))$ and transport it around the non-contractible loop γ, we end up with a different matrix $g(\gamma(s+1)) = g_\gamma g(\gamma(s))$ with some $g_\gamma \in \mathrm{SL}(2,\mathbb{R})$ (notice that g_γ appears *to the left* as $A = g^{-1}\,\mathrm{d}g$ is single-valued around γ). It is now completely straightforward to evaluate the path ordered exponential for an arbitrary curve C starting at x and ending at y (we use the same notation as in section 3.2). We have

$$U_C(y,x) = \mathcal{P}\exp\left(-\int_x^y A\right) = g(y)^{-1}g(x)\,. \tag{4.22}$$

Under a local Lorentz transformation, $U_C(y,x) \to h(y)^{-1}U_C(y,x)h(x)$. Thus, closing the loop by shifting the point y around the loop back to the base point x, we arrive at

$$U_\gamma(x) \equiv \mathcal{P}\exp\left(-\oint_\gamma A\right) = g^{-1}(x)g_\gamma g(x)\,, \tag{4.23}$$

whose trace depends only on g_γ. This trace is Lorentz invariant by construction, but it is easy to see that it is also invariant under diffeomorphisms and supersymmetry. Diffeomorphism invariance follows from the flatness of the connection A, which allows us to continuously deform the loop without changing the value of (4.22). Supersymmetry follows trivially from the invariance of A under the local supersymmetry transformations (4.7). Consequently, the Wilson loop observable needs no modification in supergravity. In passing, and already with an eye towards the discussion in section 4, we note that diffeomorphism invariance of the Wilson loop (and the other observables as well) is lost as soon as matter couplings are switched on, since the field strength then no longer vanishes, and the Wilson loop picks up extra contributions when γ is deformed (by the non-Abelian Stokes theorem). It should thus be kept in mind that pure gravity and supergravity in three dimensions represent a rather special situation, and that the results may have no relevance to the real world, which is filled with propagating and not just topological degrees of freedom.

Observables depending on the canonical variable $\Pi_a{}^i$ or, equivalently, on the dreibein form e^a, also exist, as we already know from our brief discussion of loop variables in section 3.2; the difference is that these phase space functionals are now genuine observables in the connection representation. We will write these observables in the form

$$\mathcal{O} = \oint_\gamma \omega \tag{4.24}$$

with locally Lorentz invariant closed one-forms ω on the space-like hypersurface (we will not explicitly indicate the dependence of the one-form on γ). Closure is required because we must have $\oint_C \omega = 0$ for a *contractible* closed loop C; this means that the value of $\oint_\gamma \omega$ is unaffected by continuous deformations of the loop γ, which is just another way of expressing the invariance under spatial diffeomorphisms. Invariance under local supersymmetry holds whenever the one-form varies by the exterior derivative of a single-valued function f, i.e., $\delta_\epsilon \omega = \mathrm{d}f$ with $f(\gamma(s+1)) = f(\gamma(s))$ for an arbitrary closed loop γ because then $\delta_\epsilon \oint \omega = \oint_\gamma \mathrm{d}f = 0$. For pure gravity (*not* supergravity), an appropriate one-form can be immediately deduced from the results of section 3.2; it is

$$\omega^{(1)}[\gamma;x] = \epsilon_{ij}\,\mathrm{d}x^i\, T^1[\gamma;x]^j = \operatorname{Tr} U_\gamma(\gamma(1),x)\gamma_a e^a(x) U_\gamma(x,\gamma(0)), \tag{4.25}$$

where γ is again a non-contractible closed loop along which x varies. Lorentz invariance follows because $\gamma_a e^a$ transforms according to the adjoint representation under Lorentz transformations, and closure follows from

$$\mathrm{d}\omega^{(1)}(x) = \operatorname{Tr} U_\gamma(\gamma(1),x)\gamma_a De^a(x) U_\gamma(x,\gamma(0)) = 0, \tag{4.26}$$

since $De^a = 0$ is just the Lorentz constraint in pure gravity [in the explicit computation, we use the well-known formulas $\mathrm{d}U_\gamma(\cdot,x) = U_\gamma(\cdot,x)A(x)$ and $\mathrm{d}U_\gamma(x,\cdot) = -A(x)U_\gamma(x,\cdot)$].

Unlike (4.23), however, (4.25) must be modified in supergravity, because $De^a = \frac{1}{2}\overline{\psi}\gamma^a\psi \neq 0$ and e^a is not invariant under local superymmetry anymore. We now find (with $\omega \equiv \mathrm{d}x^i\,\omega_i$)

$$\begin{aligned}\delta_\epsilon \omega_i^{(1)}(x) &= \bar{\epsilon}\gamma^a\psi_i(x)\,\mathrm{Tr}\,U_\gamma(\gamma(1),x)\gamma_a U_\gamma(x,\gamma(0)) \\ &= \bar{\epsilon}(x)U_\gamma(x,\gamma(0))U_\gamma(\gamma(1),x)\psi_i(x) \\ &\quad - \overline{\psi}_i(x)U_\gamma(x,\gamma(0))U_\gamma(\gamma(1),x)\epsilon(x) \\ &= \overline{\psi}_i(x)g^{-1}(x)g_\gamma g(x)\epsilon(x) - \bar{\epsilon}(x)g^{-1}(x)g_\gamma g(x)\psi_i(x), \end{aligned} \tag{4.27}$$

where we used the formula $U(x,\gamma(0))U(\gamma(1),x) = U_\gamma(x) = g^{-1}(x)g_\gamma g(x)$, the Fierz identity

$$\gamma^a_{\alpha\beta}\gamma_{a\gamma\delta} = \delta_{\beta\gamma}\delta_{\alpha\delta} - C^{-1}_{\gamma\alpha}C_{\beta\delta}, \tag{4.28}$$

and the relation $C^{-1}g^{\mathrm{T}}C = g^{-1}$ for $g \in \mathrm{SL}(2,\mathbb{R})$. Moreover, the one-form $\omega^{(1)}$ is no longer closed since

$$\begin{aligned}\partial_{[i}\omega_{j]}^{(1)}(x) &= -\overline{\psi}_{[i}(x)U_\gamma(x,\gamma(0))U_\gamma(\gamma(1),x)\psi_{j]}(x) \\ &= -\overline{\psi}_{[i}(x)g^{-1}(x)g_\gamma g(x)\psi_{j]}(x), \end{aligned} \tag{4.29}$$

where we used the Lorentz constraint and the Fierz identity once more. To cancel these contributions we must now construct a one-form $\omega^{(2)}$ depending on the gravitinos as well.

The desired extension is obtained by adding the one-form $\omega^{(2)}(x) \equiv \mathrm{d}x^i \times \omega_i^{(2)}(x)$ with

$$\omega_i^{(2)}(x) \equiv -\int_x^{\gamma(1)} \mathrm{d}y^j\,\overline{\psi}_j(y)U_\gamma(y,x)\psi_i(x) \tag{4.30}$$

$$-\int_{\gamma(0)}^{x} \mathrm{d}y^j\,\overline{\psi}_j(y)U_\gamma(y,\gamma(0))U_\gamma(\gamma(1),x)\psi_i(x) \tag{4.31}$$

to $\omega^{(1)}$ (note that the points x and $\gamma(0) = \gamma(1)$ are connected "along different sides" of the loop γ in the two terms). Defining $\omega \equiv \omega^{(1)} + \omega^{(2)}$, and making use of the supersymmetry constraint $D\psi = 0$, it is straightforward to show that $\mathrm{d}\omega = 0$ and $\delta_\epsilon\omega = \mathrm{d}f$, where the function

$$\begin{aligned} f(x) \equiv &\int_x^{\gamma(1)} \mathrm{d}y^j\,\overline{\psi}_j(y)U_\gamma(y,x)\epsilon(x) \\ &+ \int_{\gamma(0)}^{x} \mathrm{d}y^j\,\overline{\psi}_j(y)U_\gamma(y,\gamma(0))U_\gamma(\gamma(1),x)\epsilon(x) \end{aligned} \tag{4.32}$$

is globally defined, as is most easily seen by again expressing U_γ in terms of $g(x)$ and g_γ. We note that $\oint_\gamma \omega^{(2)} = 0$ if there exists a globally defined spinor

ϕ such that $\psi = D\phi$. This can be seen by inserting $U_\gamma(y,x) = g^{-1}(y)g(x)$ and $g(x)D_i\phi(x) = \partial_i(g\phi(x))$ into the definition of $\omega^{(2)}$ and then integrating over y, which gives a total derivative. Consequently, this observable "sees" only the topologically non-trivial part of the gravitino. This is completely analogous to the result that the holonomy is unity for trivial gauge connections, where $A = g^{-1}\mathrm{d}g$ with globally defined $g(x)$. We mention that the observables for supergravity coincide with those constructed from a "super Wilson loop" in a superspace formulation of supergravity [37], but so far they have not been explicitly written out in components.

The observables that we have constructed are therefore sensitive only to the topological excitations, of which there are only finitely many independent ones for a given surface of finite genus, and it is thus appropriate to recall how many degrees of freedom they represent. For the bosonic theory it is known that, on a surface of genus $g \geq 2$ and for an arbitrary gauge group G, the dimension of the space of non-trivial flat connections modulo gauge transformations is $(2g-2)\dim G$ [#5]. This follows from the relation $\prod \alpha_j\beta_j\alpha_j^{-1}\beta_j^{-1} = 1$ for the non-trivial homology cycles, with the corresponding relation $\prod g_{\alpha_j}g_{\beta_j}g_{\alpha_j}^{-1}g_{\beta_j}^{-1} = 1$ for the holonomies, which removes dim G degrees of freedom; another dim G degrees of freedom are removed by conjugating all holonomies with an arbitrary $\mathrm{SL}(2,\mathbb{R})$ matrix. However, one must be a little careful because this counting argument only works for generic matrices, and in certain special cases there may be extra solutions. Such is the case for genus $g = 1$ (the torus), where the dimension is 2 rank G [#6]; for $g = 0$ (the sphere) there are no non-trivial flat connections.

As for the non-trivial gravitino modes, the result is perhaps less well known, so we explain it in a little more detail. In a given background characterized by the flat connection $A = g^{-1}\mathrm{d}g$, the supersymmetry constraint is $D(A)\psi = 0$, which is equivalent to the equation (in components)

$$\partial_{[i}\big(g(x)\psi_{j]}(x)\big) = 0\,. \tag{4.33}$$

This means that, at least locally, we can always find a spinor $\phi(x)$ such that

$$g(x)\psi_j(x) = \partial_j\phi(x)\,. \tag{4.34}$$

Imposing the periodicity constraint $\psi_i(\gamma(1)) = \psi_i(\gamma(0))$ for any non-trivial cycle γ and using the previous results, we obtain the condition

$$g_\gamma^{-1}\phi(\gamma(1)) = \phi(\gamma(0)) + \phi_\gamma\,. \tag{4.35}$$

#5 Since dim $\mathrm{SL}(2,\mathbb{R}) = 3$, the dimension is $6g-6$, which suggests that the moduli space of flat $\mathrm{SL}(2,\mathbb{R})$ connections is directly related to Teichmüller space. The precise relation is explained in ref. [38].

#6 A detailed discussion of this case may be found in ref. [36].

Here ϕ_γ is a constant spinor that represents the obstruction to defining ϕ globally on the surface. In other words, this spinor is the fermionic analog of the holonomy g_γ. Just as the holonomies, the constant spinors ϕ_γ are subject to a relation that follows from the constraint on the homology cycles. To obtain it, we simply iterate (4.35) by transporting $\phi(x)$ around the homotopically trivial curve $\prod \alpha_j\beta_j\alpha_j^{-1}\beta_j^{-1}$, demanding that $\phi(x)$ return to its original value in this process. Since the spinor has two components, this removes two degrees of freedom; another two can be subtracted by noticing that (4.34) is invariant under constant shifts $\phi(x) \mapsto \phi(x) + \phi_0$, which leads to $\phi_\gamma \mapsto \phi_\gamma + (g_\gamma^{-1} - 1)\phi_0$. Altogether, we arrive at the result that the space of fermionic moduli has dimension $4g - 4$, which suggests that it is nothing but the fermionic extension of Teichmüller space (again, we have to keep in mind that the counting works only for the generic case, so the result for the torus is different).

Remarkably, the observables found here enable us to find genuine solutions to the quantum constraints as well. Namely, upon making the usual replacement of the phase space variables by operators, we need only identify a suitable "vacuum functional", which is annihilated by the constraints, and which, for the bosonic theory, is simply $\Psi_0[A] = \mathbf{1}$ (for supergravity, it is slightly more complicated, see below). Collectively denoting the observables by $\mathcal{O}^{(i)}$, we then obtain further solutions by applying these operators to the vacuum functional according to

$$\Psi[A] = \prod_i \mathcal{O}^{(i)}\Psi_0[A]. \tag{4.36}$$

This shows that, at least for gravity and supergravity in three dimensions, the quantum states are in one-to-one correspondence with the observables [22,36].

The theory is quantized in the usual way by the replacement

$$\Pi_a{}^i(\boldsymbol{x}) \longrightarrow -\mathrm{i}\,\delta/\delta A_i{}^a(\boldsymbol{x}). \tag{4.37}$$

The corresponding replacement for the gravitino is slightly more subtle: the operator realization of the Dirac brackets (4.14) necessarily breaks either manifest Lorentz or reparametrization invariance because the spinors are Majorana (self-conjugate). We take $\eta_\alpha \equiv \psi_{1\alpha}$ as the basic variable (1 is a world index); it is an anticommuting element of a Grassmann algebra. The anticommutator corresponding to the Dirac bracket is then obtained with the functional differential operator (always assumed to act from the left)

$$\psi_{2\alpha}(\boldsymbol{x}) = \mathrm{i}\,\delta/\delta\bar{\eta}_\alpha(\boldsymbol{x}) = -\mathrm{i}C_{\alpha\beta}\,\delta/\delta\eta_\beta(\boldsymbol{x}). \tag{4.38}$$

The operatorial realization of the supersymmetry generator reads

$$\mathcal{S} = D_2\eta - \mathrm{i}D_1\,\delta/\delta\bar{\eta}, \tag{4.39}$$

where we have now suppressed the spinor indices. The Lorentz constraint becomes

$$L_a = 2\mathrm{i}D_i\,\delta/\delta A_i{}^a + \bar{\eta}\gamma_a\,\delta/\delta\bar{\eta}. \tag{4.40}$$

A non-trivial solution to the quantum constraints can now be constructed by first solving the supersymmetry constraint $\mathcal{S}\Psi[A,\eta] = 0$, which is the "square root" of the WDW equation, as it is first order in the functional differential operators. Set $\eta = D_1\phi$ and define the vacuum wave functional

$$\Psi_0[A,\eta] = \exp\left(\tfrac{1}{2}\mathrm{i}\int \mathrm{d}x\,\overline{\phi}D_2D_1\phi\right). \tag{4.41}$$

Using the fact that $[D_1, D_2] = 0$ for flat connections A, it is now quite straightforward to show that (4.41) is indeed a solution. To establish invariance under local Lorentz transformations, or $L_a\Psi = 0$, we must keep in mind that there is a hidden dependence of ϕ on A which can be deduced from

$$\begin{aligned} 0 = \frac{\delta}{\delta A_1{}^a(\boldsymbol{x})}\eta(\boldsymbol{y}) &= \frac{\delta}{\delta A_1{}^a(\boldsymbol{x})}D_1\phi(\boldsymbol{y}) \\ &= D_1\frac{\delta\phi(\boldsymbol{y})}{\delta A_1{}^a(\boldsymbol{x})} + \tfrac{1}{2}\gamma_a\phi(\boldsymbol{y})\delta^{(2)}(\boldsymbol{x},\boldsymbol{y}). \end{aligned} \tag{4.42}$$

As before, further solutions can now be constructed by operating on this vacuum state with the observables given above.

As a final remark, let us note that the basic trick of generating solutions from a vacuum state by means of observables is, of course, not limited to topological theories, but can also be applied in more general circumstances. For this, however, it is first of all necessary to identify suitable observables, and this may be a difficult task as we have explained before. Even if one succeeds in finding observables, there may remain problems. Above all, one must make sure that there are no quantum anomalies in the algebra of constraints and observables (since otherwise, it may no longer be true that (4.36) solves the constraints even though Ψ_0 does). Furthermore, one should make sure that the observables form a complete set, since otherwise one may not find all solutions. A nice feature of three-dimensional pure gravity and supergravity is that neither of these problems show up, so these theories can be solved completely.

5. Matter coupled supergravity

A obvious defect of the models discussed in the previous section is the absence of physical (propagating) degrees of freedom. Although the four-dimensional theory discussed in section 2 does have propagating degrees of freedom (the helicity ± 2 states of the graviton), there it is difficult to include matter as well. The problem here is not to show that the constraints can be cast into a polynomial form but rather the fact that the whole edifice erected around the loop variables collapses as soon as any matter degrees of freedom are present; the formal solutions to the quantum constraints cease to be solutions, and it seems clear that no simple modification of the loop solutions can remedy this defect. Our aim in this section is to study theories with matter, the simplest ones being the non-linear $SL(2,\mathbb{R})/SO(2)$ non-linear σ model in three dimensions and its locally ($N = 2$) supersymmetric extension. These are the dimensionally reduced versions of pure gravity and supergravity in four dimensions, respectively (see, e.g., refs. [39,40] for further details of dimensional reduction, and ref. [41] for a recent discussion of extended supergravities in three dimensions).

Although the actual supergravity Lagrangian is quite complicated there are three essential features that may eventually enable us to make some progress. First of all, the supersymmetry constraint can be regarded as the "square root" of the WDW operator and one can hope that the corresponding quantum constraint may be easier to solve (this idea is not new; see, e.g., ref. [42] for another attempt to exploit it). We will see explicitly that the familiar ordering problems are completely absent in the supersymmetry generator; this is the analog in quantum supergravity of the result that the vacuum energy in supersymmetric theories vanishes [43]. Moreover, all other constraint equations follow from the supersymmetry constraint, provided there are no anomalies in the quantum algebra of constraints. The second new feature is the presence of the "hidden symmetries". There is a conserved Noether charge, which is an observable in that it weakly commutes with the constraints. In the quantum version of the theory, this charge at least in principle allows us to construct new solutions from old ones by repeated application of the charge operator to any given solution (as we already saw in the last section). Our intention here is to study the simplest non-trivial example, $N = 2$ supergravity in three dimensions; this theory can be alternatively obtained by dimensional reduction of simple supergravity in four dimensions [44,6]. This provides a simpler version of the canonical treatment of extended supergravities than the one given in ref. [18].

5.1. $N = 2$ SUPERGRAVITY IN THREE DIMENSIONS

In this section, we construct an $N = 2$ supergravity Lagrangian which describes the interactions of gravity and two gravitinos with one matter multiplet corresponding to the propagating helicity ± 2 and $\pm\frac{3}{2}$ states of simple supergravity in four dimensions. We start by adding a second gravitino to (4.6), so that

$$\mathcal{L}_{\text{grav}} = \epsilon^{\mu\nu\rho}\left(\tfrac{1}{4}e_\mu{}^a F_{\nu\rho a} + \tfrac{1}{2}\overline{\psi}{}^I_\mu D_\nu \psi^I_\rho\right), \tag{5.1}$$

where $I = 1,2$. To avoid cumbersome notation, we combine the two Majorana spinors ψ^I_μ into a single complex one $\psi_\mu \equiv (\psi^1_\mu + i\psi^2_\mu)/\sqrt{2}$ (with analogous definitions for the other spinors below), and rewrite the action as

$$\mathcal{L}_{\text{grav}} = \epsilon^{\mu\nu\rho}\left(\tfrac{1}{4}e_\mu{}^a F_{\nu\rho a} + \overline{\psi}_\mu D_\nu \psi_\rho\right) \tag{5.2}$$

[note that (5.1) and (5.2) differ by a total derivative]. This action is again invariant under local Lorentz rotations, diffeomorphisms and the supersymmetry transformation

$$\delta_\epsilon \psi_\mu = D_\mu \epsilon \,, \quad \delta_\epsilon e_\mu{}^a = \bar{\epsilon}\gamma^a \psi_\mu - \overline{\psi}_\mu \gamma^a \epsilon \,, \tag{5.3}$$

which corresponds to (4.7) with the real (Majorana) spinor ϵ replaced by a complex one. In addition, it is invariant under a global U(1) symmetry $\psi \mapsto e^{iq}\psi$. This extension of the $N = 1$ supergravity Lagrangian still contains no propagating degrees of freedom and, in fact, exists for arbitrary N.

To couple this system to matter (i.e. propagating degrees of freedom), we must convert the global U(1) symmetry into a local one and introduce a gauge connection Q_μ, which (in second order formalism) is a composite field made out of the matter degrees of freedom. Assigning a U(1) charge α (which will be determined shortly) to the gravitino, the covariant derivative D_μ on the gravitino reads now

$$D_\mu \psi_\nu \equiv \nabla_\mu \psi_\nu + \tfrac{1}{2} A_\mu{}^a \gamma_a \psi_\nu - i\alpha Q_\mu \psi_\nu \,. \tag{5.4}$$

Although D_μ is defined to be the full covariant derivative here, the Christoffel connection can be dropped when the derivative acts on the gravitino because of the antisymmetrization of indices, as is well known [6].

The action (5.2) with the derivative (5.4) is now invariant under the local U(1) transformations

$$\psi_\mu \mapsto e^{i\alpha q}\psi_\mu \,, \quad Q_\mu \mapsto Q_\mu + \partial_\mu q \,, \tag{5.5}$$

but since the commutator of two D_μ's gives an extra term proportional to the field strength $Q_{\mu\nu} = \partial_\mu Q_\nu - \partial_\nu Q_\mu$, $\mathcal{L}_{\text{grav}}$ is no longer invariant under local supersymmetry transformations (5.3). Instead, we find

$$\delta_\epsilon \mathcal{L}_{\text{grav}} = -i\alpha \epsilon^{\mu\nu\rho} \left[\tfrac{1}{2} \left(\overline{\psi}_\mu \epsilon - \bar{\epsilon}\psi_\mu \right) Q_{\nu\rho} + \overline{\psi}_\mu \psi_\rho \delta_\epsilon Q_\nu \right] . \tag{5.6}$$

Here the term containing $\delta_\epsilon Q_\mu$ must be kept if Q_μ is regarded as a function of the scalar fields, but can be dropped if Q_μ is treated as an independent field and its equation of motion is used. Of course, ϵ must have the same U(1) charge α as ψ, because otherwise the transformation law (5.3) would not be U(1) covariant.

Our aim is now to add a matter Lagrangian to (5.2) such that the total action is invariant both under local U(1) and local supersymmetry. It is well known that the matter sectors in (extended) supergravities are described by non-linear σ-models [45]; since we are here not interested in discussing the most general model of this type, but rather in the simplest non-trivial example, we will immediately specialize to the bosonic SL(2,$\mathbb{R}$)/SO(2) coset space σ-model as our starting point. The Lagrangian can be alternatively derived by dimensional reduction of simple supergravity in four dimensions [44,6] after a duality redefinition of the Kaluza–Klein vector (see, e.g., refs. [39,40] for details). The SO(2) $\cong$ U(1) stability subgroup of the coset space can be identified as the helicity group of the four-dimensional theory; this also explains the charge assignments of various fields below. The group SL(2,$\mathbb{R}$) has three generators Z^1,

Z^2 and Y, where Y generates the maximal compact subgroup SO(2). Their algebra reads

$$[Y,Z^1] = -2Z^2, \quad [Y,Z^2] = 2Z^1, \quad [Z^1,Z^2] = 2Y, \tag{5.7}$$

and they are normalized such that

$$\mathrm{Tr}(YY) = -2, \quad \mathrm{Tr}(Z^k Z^l) = 2\delta^{kl}, \quad \mathrm{Tr}(YZ^k) = 0. \tag{5.8}$$

The bosonic field is an element $\mathcal{V}$ of $\mathrm{SL}(2,\mathbb{R})$. Since this matrix represents three degrees of freedom, of which only two are physical, one scalar degree of freedom must be eliminated. This is the main reason for requiring the action to be invariant under *local* U(1) transformations. The combined action of the rigid $\mathrm{SL}(2,\mathbb{R})$ and the local U(1) transformations on $\mathcal{V}$ is then given by

$$\mathcal{V}(x) \mapsto g^{-1}\mathcal{V}(x)h(x), \quad g \in \mathrm{SL}(2,\mathbb{R}), \quad h(x) \in \mathrm{SO}(2). \tag{5.9}$$

The U(1) covariant derivative on $\mathcal{V}$ contains the same gauge field as (5.4); it reads

$$D_\mu \mathcal{V} \equiv \partial_\mu \mathcal{V} - \mathcal{V} Y Q_\mu . \tag{5.10}$$

An action invariant under (5.9) is

$$\mathcal{L}_{\text{boson}} \equiv -\tfrac{1}{4} e g^{\mu\nu} \; \mathrm{Tr}(\; \mathcal{V}^{-1} D_\mu \mathcal{V} \;\; \mathcal{V}^{-1} D_\nu \mathcal{V} \;). \tag{5.11}$$

At this point, we have two possibilities for treating the gauge field Q_μ. Namely, we can define Q_μ by

$$Q_\mu = -\tfrac{1}{2} \mathrm{Tr}(\; \mathcal{V}^{-1} \partial_\mu \mathcal{V} \; Y). \tag{5.12}$$

In this way, the gauge field becomes a function of the scalar fields and thus a "composite" field, whose variation under local supersymmetry is determined from the variation of the scalar fields. The other possibility is to take Q_μ as an independent field which is subsequently determined by its equation of motion, in which case there will appear extra terms bilinear in the fermionic fields on the right hand side of (5.12). This is the analog of the usual first order formulation of gravity. If Q_μ is treated as an independent field, we can drop the terms with $\delta_\epsilon Q_\mu$ in the supersymmetry variations, but must at some point invoke the Q_μ field equation (so-called 1.5 order formalism, see ref. [6]). As for the Lagrangian, the two approaches differ only in the higher order fermionic terms.

To streamline the formulas, it is useful to introduce some further notation. We define a complex matrix $Z = (Z^1 + \mathrm{i}Z^2)/\sqrt{2}$. The commutators and normalizations now read

$$[Y,Z] = 2\mathrm{i}Z, \quad [Y,Z^*] = -2\mathrm{i}Z^*, \quad [Z,Z^*] = -2\mathrm{i}Y, \tag{5.13}$$

$$\mathrm{Tr}(YY) = -2, \quad \mathrm{Tr}(ZZ^*) = 2, \quad \mathrm{Tr}(ZZ) = 0. \tag{5.14}$$

As useful abbreviations we define

$$P_\mu \equiv \tfrac{1}{2} \mathrm{Tr}(\; \mathcal{V}^{-1} D_\mu \mathcal{V} \; Z) = \tfrac{1}{2} \mathrm{Tr}(\; \mathcal{V}^{-1} \partial_\mu \mathcal{V} \; Z), \tag{5.15}$$

$$R_\mu \equiv -\tfrac{1}{2} \mathrm{Tr}(\; \mathcal{V}^{-1} D_\mu \mathcal{V} \; Y) = -\tfrac{1}{2} \mathrm{Tr}(\; \mathcal{V}^{-1} \partial_\mu \mathcal{V} \; Y) - Q_\mu , \tag{5.16}$$

which are the components of $\mathcal{V}^{-1}D_\mu\mathcal{V}$ in the directions of the three generators. With (5.14) we have

$$\mathcal{V}^{-1}D_\mu\mathcal{V} = P_\mu Z^* + P_\mu^* Z + R_\mu Y . \tag{5.17}$$

Obviously, $R_\mu = 0$ in second order formalism. In terms of these quantities the scalar Lagrangian becomes

$$\mathcal{L}_{\text{boson}} = -e g^{\mu\nu}\left(P_\mu P_\nu^* - \tfrac{1}{2}R_\mu R_\nu\right) . \tag{5.18}$$

Although the term containing R_μ has a negative sign, leading to an action that is unbounded from below, the physical spectrum and the Hamiltonian of the theory are perfectly well behaved. This is a consequence of the fact that the scalar degree of freedom associated with the Y generator is a gauge degree of freedom and hence unphysical.

Acting on (5.17) with another derivative D_ν and antisymmetrizing in the indices μ and ν, we obtain the integrability relations

$$D_\mu P_\nu = \nabla_\mu P_\nu - 2\mathrm{i}Q_\nu P_\mu , \tag{5.19}$$

$$D_{[\mu}P_{\nu]} = 2\mathrm{i}R_{[\mu}P_{\nu]} , \tag{5.20}$$

$$Q_{\mu\nu} \equiv 2\,\partial_{[\mu}Q_{\nu]} = 4\mathrm{i}\,P^*_{[\mu}P_{\nu]} - 2\,D_{[\mu}R_{\nu]} , \tag{5.21}$$

from which we conclude that P_μ has U(1) charge 2.

As the superpartner of $\mathcal{V}$ we introduce a complex fermion field χ with the usual Dirac Lagrangian, which we write in the form

$$\mathcal{L}_{\text{fermion}} \equiv \tfrac{1}{2}e\left(D_\mu\bar{\chi}\gamma^\mu\chi - \bar{\chi}\gamma^\mu D_\mu\chi\right) , \tag{5.22}$$

where the Lorentz and U(1) derivative is given by

$$D_\mu\chi \equiv \partial_\mu\chi + \tfrac{1}{2}A_\mu{}^a\gamma_a\chi - \mathrm{i}\beta Q_\mu\chi , \tag{5.23}$$

and the U(1) charge β will be adjusted in a moment.

Since no further terms containing the spin connection appear in the Lagrangian, we can now work out the field equation for $A_\mu{}^a$ that follows from $\mathcal{L}_{\text{grav}} + \mathcal{L}_{\text{boson}} + \mathcal{L}_{\text{fermion}}$. We get

$$D_{[\mu}e_{\nu]}{}^a = \overline{\psi}_{[\mu}\gamma^a\psi_{\nu]} - \tfrac{1}{2}\epsilon^{abc}e_{\mu b}e_{\nu c}\bar{\chi}\chi . \tag{5.24}$$

As is well known, this equation can be solved for the spin connection, i.e. $A_\mu{}^a$ (second order formalism). Similarly, we can use the field equation to express Q_μ in terms of the other fields; (5.12) is then replaced by

$$Q_\mu = -\tfrac{1}{2}\,\mathrm{Tr}(\,\mathcal{V}^{-1}\partial_\mu\mathcal{V}\,Y) - \mathrm{i}\alpha e^{-1}g_{\mu\nu}\epsilon^{\nu\rho\sigma}\overline{\psi}_\rho\psi_\sigma - \mathrm{i}\beta\bar{\chi}\gamma_\mu\chi . \tag{5.25}$$

This shows that R_μ no longer vanishes in the presence of fermions, but is given by a fermionic bilinear instead.

The variation of the boson $\delta_\epsilon\mathcal{V}$ under supersymmetry should be linear in χ, and that of χ linear in the derivative of the boson. The action is invariant under the transformations (5.3) and

$$\mathcal{V}^{-1}\delta_\epsilon \mathcal{V} = \bar{\chi}\epsilon\, Z + \bar{\epsilon}\chi\, Z^*, \tag{5.26}$$

$$\delta_\epsilon \chi = \gamma^\mu \epsilon\, P_\mu, \tag{5.27}$$

if we add the usual Noether term

$$\mathcal{L}_{\text{noether}} = e\, \left(\overline{\psi}_\mu \gamma^\nu \gamma^\mu \chi\, P_\nu^* + \bar{\chi}\gamma^\mu \gamma^\nu \psi_\mu P_\nu\right). \tag{5.28}$$

Demanding U(1) invariance of this term, we arrive at the relation $\beta - \alpha = 2$. Canceling the term proportional to $Q_{\mu\nu}$ in (5.6) against a similar term obtained by varying χ in (5.28) by use of (5.27) requires the integrability relation (5.21). In second order formalism, we can neglect the term containing R_μ (we can also neglect it in 1.5 order formalism as long as we are not interested in higher order fermionic terms). This completely fixes the charge assignments to $\alpha = -\frac{1}{2}$ and $\beta = \frac{3}{2}$, in accord with our expectation that the physical fermion states correspond to the helicity $\pm\frac{3}{2}$ states of simple supergravity in four dimensions.

At this point, the Lagrangian

$$\begin{aligned}\mathcal{L}'[e,\psi,\mathcal{V},\chi,A,Q] &= \mathcal{L}_{\text{grav}} + \mathcal{L}_{\text{boson}} + \mathcal{L}_{\text{fermion}} + \mathcal{L}_{\text{noether}} \\ &= \epsilon^{\mu\nu\rho}\left(\tfrac{1}{4}e_\mu{}^a F_{\nu\rho a} + \overline{\psi}_\mu D_\nu \psi_\rho\right) - e g^{\mu\nu}\left(P_\mu P_\nu^* - \tfrac{1}{2}R_\mu R_\nu\right) \\ &\quad + \tfrac{1}{2}e\left(D_\mu\bar{\chi}\gamma^\mu\chi - \bar{\chi}\gamma^\mu D_\mu\chi\right) + e\left(\overline{\psi}_\mu\gamma^\nu\gamma^\mu\chi\, P_\nu^* + \bar{\chi}\gamma^\mu\gamma^\nu\psi_\mu P_\nu\right)\end{aligned} \tag{5.29}$$

is invariant under the local supersymmetry variations (5.3), (5.26) and (5.27) modulo higher order fermionic terms. To make the Lagrangian completely invariant, we must now add quartic terms in the fermions, whose precise form will, however, depend on whether we use 1.5 or second order formalism. The latter choice has the advantage that R_μ vanishes, and that the integrability relation (5.21) assumes a simple form. However, with this choice, we are not allowed to drop terms that contain the variation of Q_μ under supersymmetry. Using (5.26), the variations of (5.12) and (5.15) yield

$$\delta_\epsilon Q_\mu = 2\mathrm{i}\,\bar{\chi}\epsilon\, P_\mu - 2\mathrm{i}\,\bar{\epsilon}\chi\, P_\mu^*, \tag{5.30}$$

$$\delta_\epsilon P_\mu = D_\mu\left(\bar{\epsilon}\chi\right), \tag{5.31}$$

and, of course, $\delta_\epsilon R_\mu = 0$. The actual calculation of the cubic fermion terms is a little tedious and not very illuminating. It turns out that we have to add the following terms to the action:

$$\mathcal{L} = \mathcal{L}' - e\,\overline{\psi}_\mu\chi\,\bar{\chi}\psi^\mu + \tfrac{1}{2}\epsilon^{\mu\nu\rho}\left(\overline{\psi}_\mu\psi_\nu\bar{\chi}\gamma_\rho\chi + \overline{\psi}_\mu\gamma_\rho\psi_\nu\,\bar{\chi}\chi\right) - \tfrac{3}{2}e\bar{\chi}\chi\,\bar{\chi}\chi\,. \tag{5.32}$$

The complete transformation laws read ($\gamma_{\mu\nu}$ is the antisymmetrized product $\gamma_{[\mu}\gamma_{\nu]}$)

$$\delta_\epsilon e_\mu{}^a = \bar{\epsilon}\gamma^a\psi_\mu - \overline{\psi}_\mu\gamma^a\epsilon\,,$$

$$\delta_\epsilon\psi_\mu = D_\mu\epsilon + \tfrac{1}{2}\mathrm{i}\epsilon R_\mu + \tfrac{1}{2}\gamma_{\mu\nu}\epsilon\,\bar{\chi}\gamma^\nu\chi\,,$$

$$\begin{aligned} \mathcal{V}^{-1}\delta_\epsilon \mathcal{V} &= \bar{\chi}\epsilon\, Z + \bar{\epsilon}\chi\, Z^*, \\ \delta_\epsilon \chi &= \gamma^\mu \epsilon\, (P_\mu - \overline{\psi}_\mu \chi)\,. \end{aligned} \tag{5.33}$$

The term proportional to R_μ in the transformation law for ψ_μ, which vanishes by (5.12), has been introduced to make the whole expression independent of Q_μ. So we have the same expressions for the transformation laws when Q_μ is treated as an independent field, i.e. in 1.5 order formalism. But there we get other higher order corrections to $\mathcal{L}'$. The total action is then given by

$$\begin{aligned} \mathcal{L} = \mathcal{L}' &- e\,\overline{\psi}_\mu \chi\, \bar{\chi} \psi^\mu + \tfrac{1}{2}\epsilon^{\mu\nu\rho}\left(\tfrac{5}{2}\overline{\psi}_\mu \psi_\nu\, \bar{\chi}\gamma_\rho \chi + \overline{\psi}_\mu \gamma_\rho \psi_\nu\, \bar{\chi}\chi \right) \\ &+ \tfrac{1}{4} e\, \overline{\psi}_{[\mu} \psi_{\nu]}\, \overline{\psi}^\mu \psi^\nu + \tfrac{15}{8} e\, \bar{\chi}\chi\, \bar{\chi}\chi\,. \end{aligned} \tag{5.34}$$

[Note that the first order formalism would also require the determination of the supersymmetry variations of the fields Q_μ and $A_\mu{}^a$. This can be done as follows. Treating these fields as independent, the variation of $\mathcal{L}$ under supersymmetry will give something proportional to the equations of motion of Q_μ and $A_\mu{}^a$. So we can define variations $\delta_\epsilon Q_\mu$ and $\delta_\epsilon A_\mu{}^a$ such that the total variation of $\mathcal{L}$ again vanishes.] The next step would be to calculate the terms of fifth order in the fermions and show that they vanish. We will not perform this final consistency check as the theory is anyhow known to be consistent.

Note that the action $\mathcal{L}$ is also invariant under local coordinate, local Lorentz and local U(1) transformations

$$\delta_q \psi_\mu = -\tfrac{1}{2}\mathrm{i} q \psi_\mu\,, \quad \mathcal{V}^{-1}\delta_q \mathcal{V} = q\, Y\,, \quad \delta_q \chi = \tfrac{3}{2}\mathrm{i} q \chi\,, \quad \delta_q Q_\mu = \partial_\mu q\,. \tag{5.35}$$

Furthermore, it is invariant under global SL(2,$\mathbb{R}$) transformations $\mathcal{V} \mapsto g^{-1}\mathcal{V}$. In second order formalism for Q_μ the associated Noether current is given by

$$\mathcal{J}^\mu = -\tfrac{1}{2}\mathcal{V}\Big\{ \Big(Z^* (P^\mu - \overline{\psi}_\nu \gamma^\mu \gamma^\nu \chi) + \text{c.c.} \Big) + \tfrac{1}{2}\mathrm{i}\Big(\epsilon^{\mu\nu\rho}\overline{\psi}_\nu \psi_\rho - 3\bar{\chi}\gamma^\mu \chi \Big) \Big\}. \tag{5.36}$$

If Q_μ is treated as an independent field, the expression simplifies to

$$\mathcal{J}^\mu = -\tfrac{1}{2} D^\mu \mathcal{V}\mathcal{V}^{-1} + \tfrac{1}{2}\mathcal{V}\Big(\overline{\psi}_\nu \gamma^\mu \gamma^\nu \chi\, Z^* + \bar{\chi}\gamma^\nu \gamma^\mu \psi_\nu\, Z \Big)\mathcal{V}^{-1}. \tag{5.37}$$

5.2. CANONICAL TREATMENT

To the Lagrangian (5.34) we will now apply the usual Dirac method to obtain the canonical momenta and the constraints. As in the last section we will denote two-dimensional curved space indices by small letters $i, j, k, \ldots$. Since we are dealing with complex spinors, we consider χ and $\bar{\chi}$ as independent. Thus the canonical configuration variables appearing in the action and their conjugated momenta are

$$\begin{matrix} e_\mu{}^a & A_\mu{}^a & \psi_\mu & \overline{\psi}_\mu & \mathcal{V} & Q_\mu & \chi & \bar{\chi} \\ p_a{}^\mu & \Pi_a{}^\mu & \bar{\pi}^\mu & \pi^\mu & \mathcal{W} & S^\mu & \bar{\lambda} & \lambda \end{matrix} \tag{5.38}$$

Here again the index μ "takes the values" t and i. It is useful to introduce some abbreviations:

$$n \equiv eg^{tt}\,, \quad nn^i \equiv eg^{ti}\,, \quad h^{ij} \equiv g^{ij} - g^{it}g^{tj}/g^{tt}\,. \tag{5.39}$$

They are functions of the dreibein only and build a two-dimensional scalar density, vector and tensor and they are essentially the lapse and shift functions introduced in section 2, and the inverse of the two-dimensional metric, respectively. We will also use the index "n" for the combination $X_n \equiv X_t + n^i X_i$.

Let us now turn to the constraint algebra. As in the last section we get some second class constraints which have to be solved by passing over from Poisson to Dirac brackets. Calculating the momenta of the space components of the dreibein, the spin connection, the fermions and the U(1) gauge field we get

$$p_a{}^i = \delta\mathcal{L}/\delta\dot{e}_i{}^a = 0\,, \qquad \Pi_a{}^i = \delta\mathcal{L}/\delta\dot{A}_i{}^a = \tfrac{1}{2}\epsilon^{ij}e_{ja}\,, \tag{5.40}$$

$$\pi^i = \delta\mathcal{L}/\delta\dot{\overline{\psi}}_i = \tfrac{1}{2}\epsilon^{ij}\psi_j\,, \qquad \overline{\pi}^i = \delta\mathcal{L}/\delta\dot{\psi}_i = -\tfrac{1}{2}\epsilon^{ij}\overline{\psi}_j\,, \tag{5.41}$$

$$\lambda = \delta\mathcal{L}/\delta\dot{\bar{\chi}} = \tfrac{1}{2}e\gamma^t\chi\,, \qquad \bar{\lambda} = \delta\mathcal{L}/\delta\dot{\chi} = \tfrac{1}{2}e\bar{\chi}\gamma^t, \tag{5.42}$$

$$S^i = \delta\mathcal{L}/\delta\dot{Q}_i = 0\,. \tag{5.43}$$

Note that $\bar{\lambda}$ is not the Dirac conjugate spinor of λ (to ensure that the combination $\dot{\chi}_\alpha\bar{\lambda}_\alpha + \dot{\bar{\chi}}_\alpha\lambda_\alpha = -\bar{\lambda}\dot{\chi} + \dot{\bar{\chi}}\lambda$ is real, $\bar{\lambda}$ must be the negative conjugate of λ). The last constraint $S^i \approx 0$ leads to another second class constraint $\delta\mathcal{L}/\delta Q_i \approx 0$, which is, of course, just the spatial component of (5.25),

$$T_i \equiv R_i - \tfrac{3}{2}\mathrm{i}\bar{\chi}\gamma_i\chi + \tfrac{1}{2}\mathrm{i}e^{-1}g_{i\nu}\epsilon^{\nu\rho\sigma}\overline{\psi}_\rho\psi_\sigma \approx 0\,. \tag{5.44}$$

The Dirac brackets are calculated in appendix B. The non-vanishing brackets containing the spin connection are

$$\begin{aligned}
\{A_i{}^a(\boldsymbol{x}), e_j{}^b(\boldsymbol{y})\}_* &= 2\epsilon_{ij}\eta^{ab}\delta^{(2)}(\boldsymbol{x},\boldsymbol{y})\,,\\
\{A_i{}^a(\boldsymbol{x}), A_j{}^b(\boldsymbol{y})\}_* &= 2n^{-1}ee^{ta}e^{tb}\epsilon_{ij}\bar{\chi}\chi\,\delta^{(2)}(\boldsymbol{x},\boldsymbol{y})\,,\\
\{A_i{}^a(\boldsymbol{x}), \chi(\boldsymbol{y})\}_* &= -n^{-1}\epsilon^{abc}e_{ib}\gamma^t\gamma_c\chi\,\delta^{(2)}(\boldsymbol{x},\boldsymbol{y})\,,\\
\{A_i{}^a(\boldsymbol{x}), \bar{\chi}(\boldsymbol{y})\}_* &= -n^{-1}\epsilon^{abc}e_{ib}\bar{\chi}\gamma_c\gamma^t\delta^{(2)}(\boldsymbol{x},\boldsymbol{y})\,,
\end{aligned} \tag{5.45}$$

and those containing only fermions are

$$\begin{aligned}
\{\chi_\alpha(\boldsymbol{x}), \bar{\chi}_\beta(\boldsymbol{y})\}_* &= -n^{-1}\gamma^t_{\alpha\beta}\delta^{(2)}(\boldsymbol{x},\boldsymbol{y})\,,\\
\{\psi_{i\alpha}(\boldsymbol{x}), \overline{\psi}_{j\beta}(\boldsymbol{y})\}_* &= \epsilon_{ij}\delta_{\alpha\beta}\delta^{(2)}(\boldsymbol{x},\boldsymbol{y})\,.
\end{aligned} \tag{5.46}$$

All other brackets of the fields $e_i{}^a$, $A_i{}^a$, χ and ψ_i vanish.

The situation for the boson $\mathcal{V}$ and its momentum $\mathcal{W}$ is slightly more complicated because $\mathcal{V}$ is an element of $\mathrm{SL}(2,\mathbb{R})$, which is not a linear space, but the subset of the vector space of all 2×2 matrices defined by $\det\mathcal{V} = 1$. So we may take $\mathcal{V}$ to be a general matrix and get additional constraints [#7]. How this is done is also shown in appendix B. By defining a matrix derivative

$$\left(\frac{\partial}{\partial \mathcal{V}}\right)_{mn} \equiv \frac{\partial}{\partial \mathcal{V}_{nm}} \quad \Rightarrow \quad \frac{\partial}{\partial \mathcal{V}} \mathrm{Tr}(A\mathcal{V}) = A, \tag{5.47}$$

we can compute the momentum of $\mathcal{V}$ (remember that $P_n = P_t + n^i P_i$):

$$\begin{aligned} \mathcal{W} &= \delta\mathcal{L}/\delta\dot{\mathcal{V}} \\ &= -\tfrac{1}{2}\left(nP_n - n\overline{\psi}_n\chi + \epsilon^{ij}\overline{\psi}_i\gamma_j\chi\right)Z^*\mathcal{V}^{-1} \\ &\quad -\tfrac{1}{2}\left(nP_n^* - n\bar{\chi}\psi_n + \epsilon^{ij}\bar{\chi}\gamma_i\psi_j\right)Z\mathcal{V}^{-1} - \tfrac{1}{2}nR_n Y\mathcal{V}^{-1}. \end{aligned} \tag{5.48}$$

The Dirac brackets of $\mathcal{V}$ and $\mathcal{W}$, also computed in appendix B, are most conveniently written in matrix form (with A, B arbitrary 2×2 matrices). Up to spatial δ-functions they read

$$\begin{aligned} \{\mathcal{V}, \mathrm{Tr}(\mathcal{W}A)\}_* &= A - \tfrac{1}{2}\mathrm{Tr}(\mathcal{V}^{-1}A)\,\mathcal{V}, \\ \{\mathrm{Tr}(\mathcal{W}A), \mathrm{Tr}(\mathcal{W}B)\}_* &= \tfrac{1}{2}\mathrm{Tr}(\mathcal{W}A)\,\mathrm{Tr}(\mathcal{V}^{-1}B) - \tfrac{1}{2}\mathrm{Tr}(\mathcal{W}B)\,\mathrm{Tr}(\mathcal{V}^{-1}A). \end{aligned} \tag{5.49}$$

After solving the second class constraints (5.40) to (5.44) we now compute the remaining (first class) constraints. These are the derivatives of $\mathcal{L}$ with respect to the Lagrange multipliers Q_t, ψ_t, $A_t{}^a$ and $e_t{}^a$. We get

$$T \equiv \delta\mathcal{L}/\delta Q_t = -\mathrm{Tr}(\mathcal{W}\mathcal{V}Y) - \tfrac{1}{2}\mathrm{i}\epsilon^{ij}\overline{\psi}_i\psi_j + \tfrac{3}{2}\mathrm{i}e\bar{\chi}\gamma^t\chi. \tag{5.50}$$

It is easy to see that $T[q] = \int qT$ is the generator of the local U(1) transformations (5.35). The Dirac brackets are

$$\begin{aligned} \{T[q], \mathcal{V}\}_* &= \mathcal{V}Yq, \\ \{T[q], \mathcal{W}\}_* &= -Y\mathcal{W}q, \\ \{T[q], \psi_i\}_* &= -\tfrac{1}{2}\mathrm{i}q\psi_i, \\ \{T[q], \chi\}_* &= \tfrac{3}{2}\mathrm{i}q\chi. \end{aligned} \tag{5.51}$$

One might expect that T has non-vanishing brackets with the spin connection since it contains χ and $e_i{}^a$, but a short calculation shows that

$$\{e\bar{\chi}\gamma^t\chi, A_i{}^a\}_* = 0. \tag{5.52}$$

It is here that we meet again with the theory defined by the Lagrangian (5.32), where Q_μ is not an independent field but given by eq. (5.12). Evaluating the canonical formalism in that case would lead to the same constraint T not by differentiating $\mathcal{L}$ with respect to Q_t but by observing that the component of $\mathcal{W}$ in the direction of Y is not free but given by the fermion terms in (5.50).

[#7] This method works equally for other non-exceptional groups, but is much more difficult to implement for the exceptional groups. That is why in ref. [18] a different parametrization was adopted.

Since Q_t itself is a Lagrange multiplier and does not appear in any constraint and because the spatial components Q_i can be eliminated by the second class constraint (5.44), we get the same algebra of first class constraints in both cases.

The Lorentz constraint is given by

$$L_a = \delta\mathcal{L}/\delta A_t{}^a = \tfrac{1}{2}\epsilon^{ij}\big(D_i e_{ja} - \overline{\psi}_i\gamma_a\psi_j\big) - \tfrac{1}{2}e e_a{}^t\bar{\chi}\chi \tag{5.53}$$

and we define the generator of local Lorentz transformation as $L[\omega^a] = \int \omega^a L_a$, which acts on the fields as

$$\begin{aligned}\{L[\omega^a], e_{ib}\}_* &= \epsilon_{bac}\omega^a e_i{}^c,\\ \{L[\omega^a], A_{ib}\}_* &= \partial_i\omega_b + \epsilon_{bac}\omega^a A_i{}^c,\\ \{L[\omega^a], \psi_i\}_* &= -\tfrac{1}{2}\omega^a\gamma_a\psi_i,\\ \{L[\omega^a], \chi\}_* &= -\tfrac{1}{2}\omega^a\gamma_a\chi\,.\end{aligned} \tag{5.54}$$

Taking the derivative of $\mathcal{L}$ with respect to $\overline{\psi}_t$ we get the supersymmetry constraint

$$\begin{aligned}\mathcal{S} = \delta\mathcal{L}/\delta\overline{\psi}_t &= -\chi\,\mathrm{Tr}(\mathcal{W}\mathcal{V}Z^*) + \epsilon^{ij}\big(D_i\psi_j + \tfrac{1}{2}\mathrm{i}R_i\psi_j\big)\\ &\quad - \epsilon^{ij}e_i{}^a\big(\gamma_a\chi P_j^* + \tfrac{1}{2}\gamma_{ab}\psi_j\bar{\chi}\gamma^b\chi\big),\end{aligned} \tag{5.55}$$

where we have used the constraint (5.44) to obtain the term proportional to R_i. The generator of local supersymmetry transformation is defined by $\mathcal{S}[\epsilon] = \int \bar{\epsilon}\mathcal{S} + \overline{\mathcal{S}}\epsilon$ and it is straightforward to verify that for any field ϕ we have $\{\mathcal{S}[\epsilon],\phi\}_* = \delta_\epsilon\phi$ with the transformation defined by (5.33), but since some of them contain time derivatives, they are only equal modulo the equations of motion; e.g., we have for the matter fields

$$\begin{aligned}\{\mathcal{S}[\epsilon],\mathcal{V}\}_* &= \bar{\epsilon}\chi\,\mathcal{V}Z^* + \bar{\chi}\epsilon\mathcal{V}Z\,,\\ \{\mathcal{S}[\epsilon],\mathcal{W}\}_* &= -\bar{\epsilon}\chi\,Z^*\mathcal{W} - \bar{\chi}\epsilon Z\mathcal{W} - \tfrac{1}{2}\epsilon^{ij}\mathcal{V}^{-1}D_i\\ &\quad\times\Big(\tfrac{1}{2}(\overline{\psi}_j\epsilon - \bar{\epsilon}\psi_j)\,\mathcal{V}Y\mathcal{V}^{-1} + \bar{\epsilon}\gamma_j\chi\,\mathcal{V}Z\mathcal{V}^{-1} - \bar{\chi}\gamma_j\epsilon\,\mathcal{V}Z^*\mathcal{V}^{-1}\Big),\\ \{\mathcal{S}[\epsilon],P_i\}_* &= (D_i + 2iR_i)\,\bar{\epsilon}\chi\,,\\ \{\mathcal{S}[\epsilon],R_i + Q_i\}_* &= 2\mathrm{i}P_i\bar{\chi}\epsilon - 2\mathrm{i}P_i^*\bar{\epsilon}\chi\,,\\ \{\mathcal{S}[\epsilon],\chi\}_* &= h^{ij}\gamma_i\epsilon\,\widehat{P}_j - n^{-1}\gamma^t\epsilon\,\widetilde{P},\end{aligned} \tag{5.56}$$

where

$$\widehat{P}_i = P_i - \overline{\psi}_i\chi\,, \qquad \widetilde{P} = \mathrm{Tr}(\mathcal{W}\mathcal{V}Z) + \epsilon^{ij}\overline{\psi}_i\gamma_j\chi\,. \tag{5.57}$$

Finally we will compute the diffeomorphism and Hamiltonian constraints, i.e., we must evaluate $\mathcal{H}_a = \delta\mathcal{L}/\delta e_t{}^a$. It takes a little further calculation to construct the generator of diffeomorphisms and the WDW constraint from $\mathcal{H}_a$. A slightly quicker method for the diffeomorphism generator is to require that its Dirac bracket with any field should yield the corresponding Lie derivative on this field, viz.,

$$\{\mathcal{D}[\zeta^k], e_i{}^a\}_* = \partial_i\zeta^k e_k{}^a + \zeta^k\partial_k e_i{}^a. \tag{5.58}$$

With $\mathcal{D}[\zeta^k] = \int \zeta^k \mathcal{D}_k$, this leads to

$$\mathcal{D}_k = -\operatorname{Tr}(\mathcal{W}\partial_k \mathcal{V}) + \tfrac{1}{2}\epsilon^{ij}(\partial_i A_j{}^a e_{ka} + A_k{}^a \partial_i e_{ja}) - \tfrac{1}{2}e(\partial_k \bar{\chi}\gamma^t \chi - \bar{\chi}\gamma^t \partial_k \chi) + \epsilon^{ij}(\partial_i \overline{\psi}_j \psi_k + \overline{\psi}_k \partial_i \psi_j). \tag{5.59}$$

As shown in appendix C this can be obtained from $\mathcal{H}_a$ by

$$\mathcal{D}_k = e_k{}^a \mathcal{H}_a + Q_k T + A_k{}^a L_a + \overline{\psi}_k \mathcal{S} + \overline{\mathcal{S}}\psi_k \,. \tag{5.60}$$

Subtracting these other constraints from $\mathcal{D}_k$ replaces the normal derivatives by the (super)covariant ones, i.e., we can define a U(1) and Lorentz covariant diffeomorphism constraint

$$\mathcal{D}_k - Q_k T - A_k{}^a L_a = -\operatorname{Tr}(\mathcal{W} D_k \mathcal{V}) + \tfrac{1}{4} e_k{}^a \epsilon^{ij} F_{ija} - \tfrac{1}{2}e(D_k \bar{\chi}\gamma^t \chi - \bar{\chi}\gamma^t D_k \chi) + \epsilon^{ij}(D_i \overline{\psi}_j \psi_k + \overline{\psi}_k D_i \psi_j), \tag{5.61}$$

which generates extra U(1) and Lorentz rotations with parameters $-\zeta^k Q_k$ and $-\zeta^k A_k{}^a$, respectively, and which gives the correct result only weakly.

The Hamiltonian constraint, as defined in appendix C, is

$$\begin{aligned}\mathcal{H} = {} & \widetilde{P}\widetilde{P}^* - neh^{ij}\Big(\widehat{P}_i \widehat{P}_j^* - \widetilde{D}_i \bar{\chi}\gamma_j \chi + \bar{\chi}\gamma_i \widetilde{D}_j \chi\Big) \\ & + e\epsilon^{ij}\Big(\tfrac{1}{4}e^{ta}F_{ija} + (P_i - \tfrac{1}{2}\overline{\psi}_i \chi)\bar{\chi}\gamma^t \psi_j - (P_i^* - \tfrac{1}{2}\bar{\chi}\psi_i)\overline{\psi}_j \gamma^t \chi\Big) \\ & - \tfrac{3}{2}e^2 \bar{\chi}\chi\bar{\chi}\chi\,,\end{aligned} \tag{5.62}$$

where $\widehat{P}_i$ and $\widetilde{P}$ are given by (5.57). As also seen in appendix C, the diffeomorphism and Hamiltonian constraints defined in this way are equivalent to the "true" constraints $\mathcal{H}_a$ if and only if the dreibein itself is not degenerate; otherwise a solution of (5.59) and (5.62) will not in general be a solution of the theory defined by the Lagrangian (5.34), which contains the inverse dreibein, too. Evidently, all constraints are polynomial in terms of the canonical variables that we have chosen.

Finally we have to express the conserved charge of the current (5.37) in terms of the canonical variables, since this is expected to be an observable in the sense of Dirac, i.e., it should weakly commute with all constraints. We have

$$\mathcal{Q} = \int \mathrm{d}^2x\, e\, \mathcal{J}^t = \int \mathrm{d}^2x\, \mathcal{V}\mathcal{W}\,. \tag{5.63}$$

One can obtain this simple expression without using the current (5.37); consider a space–time dependent SL(2,$\mathbb{R}$) transformation with parameter $g^{-1}(x)$. Then the current is given by

$$\delta S = \int \mathrm{d}^2x \mathrm{d}t\, e \operatorname{Tr}(\mathcal{J}^\mu \partial_\mu g^{-1})\,. \tag{5.64}$$

On the other hand, since only $\mathcal{V}$ transforms under SL(2,$\mathbb{R}$), we have

$$\delta S = \int \mathrm{d}t\, \delta\mathcal{L} = \int \mathrm{d}^2x\, \mathrm{d}t \operatorname{Tr}\left(\frac{\delta\mathcal{L}}{\delta\mathcal{V}} g^{-1}\mathcal{V} + \frac{\delta\mathcal{L}}{\delta\dot{\mathcal{V}}}\partial_t(g^{-1}\mathcal{V})\right). \tag{5.65}$$

From this we infer [#8] that the time component of the current must be $\mathcal{VW}$.

5.3. ALGEBRA OF CHARGES AND CONSTRAINTS

A final check for our calculation is to show that the constraints are indeed first class, i.e., that their Dirac brackets vanish weakly. The U(1), Lorentz and diffeomorphism generators form the usual subalgebra, which we just write down for completeness:

$$\begin{aligned}\{\mathcal{D}[\zeta^k],\mathcal{D}[\tilde{\zeta}^k]\}_* &= \mathcal{D}[\tilde{\zeta}^l\partial_l\zeta^k - \zeta^l\partial_l\tilde{\zeta}^k],\\ \{L[\omega^a],\mathcal{D}[\zeta^k]\}_* &= L[\zeta^k\partial_k\omega^a],\\ \{T[q],\mathcal{D}[\zeta^k]\}_* &= T[\zeta^k\partial_k q],\\ \{L[\omega^a],L[\omega^a]\}_* &= L[\epsilon^{abc}\omega_b\omega_c].\end{aligned} \tag{5.66}$$

Their brackets with $\mathcal{S}$ may be obtained by using that $\mathcal{S}$ is a spinor density with U(1) charge $-\frac{1}{2}$, which yields

$$\begin{aligned}\{\mathcal{D}[\zeta^k],\mathcal{S}[\epsilon]\}_* &= \mathcal{S}[-\zeta^k\partial_k\epsilon],\\ \{L[\omega^a],\mathcal{S}[\epsilon]\}_* &= \mathcal{S}[\tfrac{1}{2}\omega^a\gamma_a\epsilon],\\ \{T[q],\mathcal{S}[\epsilon]\}_* &= \mathcal{S}[\tfrac{1}{2}\mathrm{i}q\epsilon].\end{aligned} \tag{5.67}$$

To get the brackets with $\mathcal{H}$ one uses that $\mathcal{H}$ is a U(1) and Lorentz scalar density of weight 2. So it commutes with L and T and we have

$$\{\mathcal{D}[\zeta^k],\mathcal{H}\}_* = \zeta^k\partial_k\mathcal{H} + 2\partial_k\zeta^k\mathcal{H}. \tag{5.68}$$

A more complicated computation shows that

$$\begin{aligned}\{\mathcal{S}[\epsilon],\mathcal{S}[\tilde{\epsilon}]\}_* = \int \mathrm{d}^2x\,&(\bar{\epsilon}\gamma^a\tilde{\epsilon} - \bar{\tilde{\epsilon}}\gamma^a\epsilon)\mathcal{H}_a + \mathrm{i}(\bar{\tilde{\epsilon}}\chi\bar{\chi}\epsilon - \bar{\epsilon}\chi\bar{\chi}\tilde{\epsilon})T\\ &+ \epsilon^{abc}(\bar{\tilde{\epsilon}}\gamma_a\epsilon - \bar{\epsilon}\gamma_a\tilde{\epsilon})\bar{\chi}\gamma_b\chi\,L_c.\end{aligned} \tag{5.69}$$

This vanishes again weakly, but it is not the generator of any combination of transformations defined above, as expected since we did not introduce auxiliary fields to close the algebra of supersymmetry transformations. The crucial point when passing over to the quantum theory is, of course, that all the "structure functions" in (5.69) appear to the left of the constraints.

What remains now is to show that the charge (5.63) indeed commutes (at least weakly) with all the constraints. By using the Dirac brackets of $T[q]$ and $L[\omega^a]$ given in (5.51) and (5.54) it is immediately clear the $\mathcal{VW}$ is U(1) and Lorentz invariant, and thus $\{T[q],\mathcal{Q}\}_* = \{L[\omega^a],\mathcal{Q}\}_* = 0$. Under supersymmetry

[#8] Note that this is not immediately clear because one might have to integrate by parts some time derivatives to proceed from (5.65) to (5.64). This would happen if $\mathcal{L}$ changes under the global symmetry by a time derivative, which is not the case here.

$\mathcal{V}\mathcal{W}$ changes by a total derivative, as can been seen from (5.56). So being an integral over the spatial manifold $\mathcal{Q}$ commutes with $\mathcal{S}[\epsilon]$, and, of course, also with the generator of diffeomorphisms $\mathcal{D}[\zeta^k]$.

To obtain the bracket with $\mathcal{H}$, we have seen already in appendix B that P_i and $R_i + Q_i$ commute with the charge $\mathcal{Q}$. Evaluating the bracket $\{\mathrm{Tr}(\mathcal{W}\mathcal{V}Z), \mathcal{V}\mathcal{W}\}_*$ using (5.49) also gives zero, which leads to $\{\widetilde{P}, \mathcal{Q}\}_* = 0$. Since $\mathcal{V}$ and $\mathcal{W}$ enter into $\mathcal{H}$ only via $\widetilde{P}$, P_i and $R_i + Q_i$, we conclude that $\{\mathcal{H}, \mathcal{Q}\}_* = 0$. This establishes our claim that $\mathcal{Q}$ is indeed a "physical observable" in the sense of Dirac, i.e., it commutes weakly[#9] with all the first class constraints. In the quantized theory and in the absence of anomalies in the algebra of constraints and charges, the corresponding operator maps physical states into physical states and can therefore be used to generate new solutions of the WDW equation from old ones.

We conclude this section (and our lectures) with some remarks concerning the quantized theory, where many open questions remain. The first step here is to find an operator representation of the algebra defined by (5.45), (5.46), (5.49). Let us first have a look at the brackets of $\mathcal{V}$ and $\mathcal{W}$. Since all the constaints depend on $\mathcal{W}$ only via the composite fields[#10] $\mathrm{Tr}(\mathcal{W}\mathcal{V}Z)$, $\mathrm{Tr}(\mathcal{W}\mathcal{V}Z^*)$ and $\mathrm{Tr}(\mathcal{W}\mathcal{V}Y)$, it is sufficient to give an operator representation for them. The Dirac brackets of these fields are given by (again we do not explicitly write out the dependence of the δ functions on the spatial coordinates)

$$\begin{aligned}\{\mathrm{Tr}(\mathcal{W}\mathcal{V}A), \mathrm{Tr}(\mathcal{W}\mathcal{V}B)\}_* &= \mathrm{Tr}(\mathcal{W}\mathcal{V}[B,A]), \\ \{\mathcal{V}, \mathrm{Tr}(\mathcal{W}\mathcal{V}A)\}_* &= \mathcal{V}A. \end{aligned} \tag{5.70}$$

A suitable operator representation is therefore

$$\mathcal{V} \mapsto \mathcal{V}, \quad \mathrm{Tr}(\mathcal{W}\mathcal{V}A) \mapsto \mathrm{Tr}(\mathrm{i}\mathcal{V}A\,\delta/\delta\mathcal{V}). \tag{5.71}$$

These operators satisfy

$$\begin{aligned}[\mathrm{Tr}(\mathrm{i}\mathcal{V}A\,\delta/\delta\mathcal{V}), \mathrm{Tr}(\mathrm{i}\mathcal{V}B\,\delta/\delta\mathcal{V})] &= (-\mathrm{i})\,\mathrm{Tr}(\mathrm{i}\mathcal{V}[B,A]\,\delta/\delta\mathcal{V}), \\ [\mathcal{V}, \mathrm{Tr}(\mathrm{i}\mathcal{V}A\,\delta/\delta\mathcal{V})] &= (-\mathrm{i})\mathcal{V}A. \end{aligned} \tag{5.72}$$

It is equally straightforward to find an operator representation for the gravitinos, as they do not mix with any other fields:

$$\overline{\psi}_i \mapsto \overline{\psi}_i, \qquad \psi_i \mapsto -\mathrm{i}\epsilon_{ik}\,\delta/\delta\overline{\psi}_k. \tag{5.73}$$

The commutator replacing (5.46) reads

$$[-\mathrm{i}\epsilon_{ik}\,\delta/\delta\overline{\psi}_{k\alpha}, \overline{\psi}_{j\beta}] = (-\mathrm{i})\epsilon_{ij}\delta_{\alpha\beta}. \tag{5.74}$$

[#9] In this case it commutes even strongly, which is a consequence of using first order formalism for Q_μ. See ref. [18], where second order formalism is used and the corresponding commutators only vanish modulo the constraints.

[#10] Indeed we have $\mathcal{W} = \frac{1}{2}\mathrm{Tr}(\mathcal{W}\mathcal{V}Z)Z^*\mathcal{V}^{-1} + \frac{1}{2}\mathrm{Tr}(\mathcal{W}\mathcal{V}Z^*)Z\mathcal{V}^{-1} - \frac{1}{2}\mathrm{Tr}(\mathcal{W}\mathcal{V}Y)Y\mathcal{V}^{-1}$ since $\mathrm{Tr}(\mathcal{W}\mathcal{V})$ vanishes by a second class constraint.

Observe that this representation is even simpler than the one used in section 4.2, because there is no need to give up manifest covariance as the fermions are complex.

For the spin connection, the dreibein and the fermion field the situation becomes more complicated. Problems are mainly caused by the new contributions on the right hand side of (5.45). So, for instance, we can no longer represent $A_i{}^a$ by a multiplication operator. To find suitable operators obeying (5.45), we must therefore search for combinations of $A_i{}^a$ and χ that split into two pairs of canonically conjugate fields. A possible ansatz is to take the *complex* connection

$$A_i^{\pm a} \equiv A_i{}^a \pm \epsilon^{abc} e_{ib} \bar{\chi} \gamma_c \chi \,, \tag{5.75}$$

whose components commute with each other,

$$\{A_i^{+a}, A_j^{+b}\}_* = \{A_i^{-a}, A_j^{-b}\}_* = 0 \,, \tag{5.76}$$

and which obey

$$\{A_i^{+a}, \chi\}_* = \{A_i^{-a}, \bar{\chi}\}_* = 0 \,. \tag{5.77}$$

Unfortunately, the brackets of A_i^{+a} with $\bar{\chi}$ and vice versa do not vanish, nor does A_i^{+a} commute with A_j^{-b}. Nevertheless it is worth observing that, by using the new connection, the supersymmetry constraint simplifies to

$$\mathcal{S} = -\chi \,\mathrm{Tr}(\mathcal{W}\mathcal{V}Z^*) + \epsilon^{ij} \widetilde{D}_i^+ \psi_j - \epsilon^{ij} \gamma_i \chi \, P_j^* \,, \tag{5.78}$$

where

$$\widetilde{D}_i^+ \psi_j = \big(\partial_i + \tfrac{1}{2} A_i^{+a} \gamma_a + \tfrac{1}{2}\mathrm{i}(Q_i + R_i)\big)\psi_j \tag{5.79}$$

is the covariant derivative with the spin connection replaced by A_i^{+a}. This explicit realization of (one half of) the canonical supersymmetry generator shows that there are no ordering ambiguities!

We could now go ahead and try to solve one half of the quantum supersymmetry constraint

$$\mathcal{S}(\boldsymbol{x})\Psi = 0 \,. \tag{5.80}$$

However, there is still the other half of the supersymmetry constraint, for which our operator representation does not work, so that even if we succeed in solving this equation, we could not claim to have solved the full WDW equation. Perhaps the resolution of this problem is the same as in (super)string theory: there, one imposes only half of the Virasoro constraints (corresponding to the Virasoro generators L_m with $m \geq 0$) on the physical states, whereas the hermitean conjugate operators (for which $m < 0$) need not annihilate them.

Appendix A

In this appendix, we present a quick proof that the crucial Poisson bracket (3.4) indeed vanishes. This proof is also given in Ashtekar's book [2] for a different choice of canonical variables. First we calculate the Poisson bracket (3.5) using

$$\begin{aligned}\{p_{ma}, p_{nb}\} &= \{e_{ma}p, e_{nb}p\} = 0\,, \\ \{p_{ma}, e_{nb}p\} &= e_{nb}p_{ma} - e_{mb}e_{na}p\,, \\ \{e^{-1}, p_{nb}\} &= e^{-1}e_{nb}\,, \\ \{e^{-1}, e_{nb}p\} &= -3e^{-1}e_{nb}\,. \end{aligned} \tag{A.1}$$

We get

$$\{\widehat{p}_{ma}, \widehat{p}_{nb}\} = (1 + 2\beta)\,(p_{nb}e_{ma} - p_{ma}e_{nb})\,. \tag{A.2}$$

So this will only vanish for $\beta = -\frac{1}{2}$. Next we define a functional

$$G \equiv -\frac{1}{8}\int \mathrm{d}x\, e\epsilon_{abc}\Omega_{abc} = -\frac{1}{4}\int \mathrm{d}x\,\epsilon^{mnr}e_{mc}\partial_n e_{rc} \tag{A.3}$$

and calculate the Poisson bracket of this functional with $\widehat{p}_{ma}$:

$$\begin{aligned}\{p_{mc}, G\} &= \tfrac{1}{2}e\,\epsilon_{abc}e_{md}\Omega_{abd}\,, \\ \{pe_{mc}, G\} &= \tfrac{1}{2}e\,\epsilon_{abd}e_{mc}\Omega_{abd} \\ \Rightarrow\quad \{\widehat{p}_{mc}, G\} &= \tfrac{1}{2}(\epsilon_{abc}e_{md} - \tfrac{1}{2}\epsilon_{abd}e_{mc})\Omega_{abd}\,. \end{aligned} \tag{A.4}$$

We have to show that this is equal to

$$-\tfrac{1}{2}\epsilon_{abc}\omega_{mab} = -\tfrac{1}{4}\epsilon_{abc}e_{md}\,(\Omega_{dab} + \Omega_{bda} - \Omega_{abd})\,. \tag{A.5}$$

By taking the difference of the last two equations and renaming some indices we find

$$\{\widehat{p}_{mc}, G\} + \tfrac{1}{2}\epsilon_{abc}\omega_{mab} = \tfrac{1}{4}\,(-\epsilon_{bcd}e_{ma} + \epsilon_{acd}e_{mb} - \epsilon_{abd}e_{mc} + \epsilon_{abc}e_{md})\,\Omega_{abd}\,. \tag{A.6}$$

The term inside the parentheses is totally skew symmetric in the four three-dimensional indices a, b, c and d. So we end up with

$$\{\widehat{p}_{mc}, G\} = -\tfrac{1}{2}\epsilon_{abc}\omega_{mab}\,. \tag{A.7}$$

Appendix B

In this appendix, we calculate the Dirac brackets of $N = 2$ supergravity in three dimensions. The second class constraints are given by eqs. (5.40) to (5.44). We include two further constraints on the matrix $\mathcal{V}$ and its conjugate momentum $\mathcal{W}$ so that we can treat them as general 2×2-matrices, i.e., we can define the momentum as

$$\mathcal{W}_{mn} = \delta\mathcal{L}/\delta\dot{\mathcal{V}}_{nm}\,, \tag{B.1}$$

which is just (5.48) in matrix notation. The Poisson bracket of $\mathcal{V}$ and $\mathcal{W}$ is

$$\{\mathcal{V}_{mn}, \mathcal{W}_{pq}\} = \delta_{mq}\delta_{pn} \quad\Rightarrow\quad \begin{aligned}\{\mathrm{Tr}(\mathcal{V}A), \mathrm{Tr}(\mathcal{W}B)\} &= \mathrm{Tr}(AB)\,, \\ \{\det\mathcal{V}, \mathrm{Tr}(\mathcal{W}B)\} &= \mathrm{Tr}(\mathcal{V}^{-1}B)\det\mathcal{V}\,, \end{aligned} \tag{B.2}$$

for some matrices A and B. Here as in the following we will not write down the space dependence of the fields and all the brackets are to be multiplied by the spatial δ-function. The additional constraints ensure that $\mathcal{V} \in \mathrm{SL}(2,\mathbb{R})$ and that $\dot{\mathcal{V}}$ is tangent to $\mathrm{SL}(2,\mathbb{R})$. The full set of second class constraints now read

$$\begin{aligned}
P_a^i &\equiv p_a{}^i \approx 0\,,\\
Z_a^i &\equiv \Pi_a{}^i - \tfrac{1}{2}\epsilon^{ij}e_{ja} \approx 0\,,\\
\Lambda &\equiv \lambda - \tfrac{1}{2}e\,\gamma^t\chi \approx 0\,,\\
\overline{\Lambda} &\equiv -\bar{\lambda} + \tfrac{1}{2}e\,\bar{\chi}\gamma^t \approx 0\,,\\
\Gamma^i &\equiv -\pi^i + \tfrac{1}{2}\epsilon^{ij}\psi_j \approx 0\,,\\
\overline{\Gamma}^i &\equiv \bar{\pi}^i + \tfrac{1}{2}\epsilon^{ij}\overline{\psi}_j \approx 0\,,\\
S^i &\approx 0\,,\\
T_i &\equiv R_i - \tfrac{3}{2}\mathrm{i}\bar{\chi}\gamma_i\chi + \tfrac{1}{2}e^{-1}g_{i\nu}\epsilon^{\nu\rho\sigma}\overline{\psi}_\rho\psi_\sigma \approx 0\,,\\
V &\equiv \det\mathcal{V} - 1 \approx 0\,,\\
W &\equiv \mathrm{Tr}(\mathcal{W}\mathcal{V}) \approx 0\,.
\end{aligned} \tag{B.3}$$

The Dirac brackets [8] are defined by

$$\{A,B\}_* = \{A,B\} - \sum_{K,L}\{A,K\}C(K,L)\{L,B\}\,, \tag{B.4}$$

where $K, L, M, \ldots$ stand for the above constraints and the coefficients $C(K,L)$ are (at least weakly) given by

$$\sum_L C(K,L)\{L,M\} = \delta(K,M)\,, \tag{B.5}$$

i.e., $C(.,.)$ is the inverse matrix of $\{.,.\}$. By $\delta(K,M)$ we mean 1 if K and M are the same constraints and 0 otherwise, e.g., $\delta(Z_a^i, Z_b^j) = \delta_b^a\delta_i^j$ (observe the position of indices).

By writing out eq. (B.5) for fixed M and using part of the Poisson algebra of the constraints we get the following formulas, which can be used to compute all components of $C(.,.)$:

$$\begin{aligned}
C(K,P_a^i) &= -2\epsilon_{ij}\eta^{ab}\delta(K,Z_b^j)\,,\\
C(K,T_i) &= -\delta(K,S^i)\,,\\
C(K,W) &= -\tfrac{1}{2}\delta(K,V)\,,\\
C(K,V) &= \tfrac{1}{2}\big(\delta(K,W) - C(K,T_i)\{T_i,W\}\big)\,,\\
C(K,\Lambda_\alpha) &= -n^{-1}\gamma^t_{\beta\alpha}\big(\delta(K,\overline{\Lambda}_\beta) - C(K,P_a^i)\{P_a^i,\overline{\Lambda}_\beta\} - C(K,T_i)\{T_i,\overline{\Lambda}_\beta\}\big)\,,\\
C(K,\overline{\Lambda}_\alpha) &= -n^{-1}\gamma^t_{\alpha\beta}\big(\delta(K,\Lambda_\beta) - C(K,P_a^i)\{P_a^i,\Lambda_\beta\} - C(K,T_i)\{T_i,\Lambda_\beta\}\big)\,,\\
C(K,\Gamma^i) &= \epsilon_{ji}\big(\delta(K,\overline{\Gamma}^j) - C(K,T_k)\{T_k,\overline{\Gamma}^j\}\big)\,,
\end{aligned}$$

$$
\begin{aligned}
C(K,\overline{\Gamma}^i) &= \epsilon_{ij}\big(\delta(K,\Gamma^j) - C(K,T_k)\{T_k,\Gamma^j\}\big)\,,\\
C(K,Z^i_a) &= -2\epsilon_{ij}\eta^{ab}\big(\delta(K,P^j_b) - C(K,\Lambda_\alpha)\{\Lambda_\alpha,P^j_b\}\\
&\quad - C(K,\overline{\Lambda}_\alpha)\{\overline{\Lambda}_\alpha,P^j_b\} - C(K,T_k)\{T_k,P^j_b\}\big)\,,\\
C(K,S^i) &= \delta(K,T_i) - C(K,W)\{W,T_i\} - C(K,\Lambda_\alpha)\{\Lambda_\alpha,T_i\}\\
&\quad - C(K,\overline{\Lambda}_\alpha)\{\overline{\Lambda}_\alpha,T_i\} - C(K,\Gamma^k_\alpha)\{\Gamma^k_\alpha,T_i\}\\
&\quad - C(K,\overline{\Gamma}^k_\alpha)\{\overline{\Gamma}^k_\alpha,T_i\} - C(K,P^j_b)\{P^j_b,T_i\}\,.
\end{aligned}
\tag{B.6}
$$

Note that $C(.,.)$ is antisymmetric only if both entries are bosonic, otherwise it is symmetric. We can use the constraints (B.3) to express any quantity in terms of $A_i{}^a$, e^a_i, χ, $\bar{\chi}$, ψ_i, $\overline{\psi}_i$, $\mathcal{V}$ and $\mathcal{W}$, so we have to evaluate the Dirac brackets of these field only. As an example we calculate the Dirac bracket of $A_k{}^c$ with the other fields:

$$
\begin{aligned}
\{A_i{}^a,e_j{}^b\}_* &= -\{A_i{}^a,Z^k_c\}C(Z^k_c,P^l_d)\{P^l_d,e_j{}^b\}\\
&= C(Z^i_a,P^j_b) = 2\epsilon_{ij}\eta^{ab}\,.
\end{aligned}
\tag{B.7}
$$

The Dirac bracket of two spin connections does not vanish but gives

$$
\begin{aligned}
\{A_i{}^a,A_j{}^b\}_* &= C(Z^i_a,Z^j_b)\\
&= 2\epsilon_{jk}\eta^{bc}\big(C(Z^i_a,\Lambda_\alpha)\{\Lambda_\alpha,P^k_c\} + C(Z^i_a,\overline{\Lambda}_\alpha)\{\overline{\Lambda}_\alpha,P^k_c\}\big)\\
&= 2n^{-1}ee^{ta}e^{tb}\epsilon_{ij}\bar{\chi}\chi\,.
\end{aligned}
\tag{B.8}
$$

Because this contains the fermion, there must be a non-vanishing bracket of $A_i{}^a$ with χ to ensure that the brackets obey the Jacobi identity,

$$
\begin{aligned}
\{A_i{}^a,\chi_\alpha\}_* &= -\{A_i{}^a,Z^j_b\}C(Z^j_b,\overline{\Lambda}_\beta)\{\overline{\Lambda}_\beta,\chi_\alpha\} = -C(Z^i_a,\overline{\Lambda}_\alpha)\\
&= -n^{-1}\epsilon^{abc}e_{ib}(\gamma^t\gamma_c\chi)_\alpha\,.
\end{aligned}
\tag{B.9}
$$

The brackets of $A_i{}^a$ with ψ_j, $\mathcal{V}$ or $\mathcal{W}$ vanish since the components $C(Z^a_i,K)$ vanish for $K = \Gamma^j$, V, W or T_j, so altogether we get (5.45). In the same way one can see that χ has non-vanishing brackets only with $\bar{\chi}$ and $A_i{}^a$ and ψ_i only with $\overline{\psi}_j$, giving (5.46). What remains are the brackets of $\mathcal{V}$ and $\mathcal{W}$. Since the only constraint that does not commute with $\mathcal{V}$ is W, we have $\{\mathcal{V},\mathcal{V}\}_* = 0$ and

$$
\begin{aligned}
\{\mathcal{V}_{mn},\mathcal{W}_{pq}\}_* &= \{\mathcal{V}_{mn},\mathcal{W}_{pq}\} - \{\mathcal{V}_{mn},W\}C(W,V)\{V,\mathcal{W}_{pq}\}\\
&= \delta_{mq}\delta_{pn} - \tfrac{1}{2}\mathcal{V}_{mn}\mathcal{V}^{-1}_{pq}\,,
\end{aligned}
\tag{B.10}
$$

$$
\begin{aligned}
\{\mathcal{W}_{mn},\mathcal{W}_{pq}\}_* &= -\{\mathcal{W}_{mn},V\}C(V,W)\{W,\mathcal{W}_{pq}\}\\
&\quad - \{\mathcal{W}_{mn},W\}C(W,V)\{V,\mathcal{W}_{pq}\}\\
&= \tfrac{1}{2}\mathcal{W}_{mn}\mathcal{V}^{-1}_{pq} - \tfrac{1}{2}\mathcal{W}_{pq}\mathcal{V}^{-1}_{mn}\,.
\end{aligned}
\tag{B.11}
$$

Written in matrix notation we get (5.49). From these basic brackets one can compute the brackets of the composite fields P_i, P^*_i and $R_i + Q_i$ with $\mathcal{W}$, which one needs to obtain the algebra of the constraints. A straightforward calculation shows that

$$\{\mathrm{Tr}(\mathcal{V}^{-1}\partial_i\mathcal{V}\,A)(\boldsymbol{x}),\mathcal{W}(\boldsymbol{y})\}_* = -\mathcal{V}^{-1}(\boldsymbol{y})\frac{\partial}{\partial y^i}\left(\mathcal{V}A\mathcal{V}^{-1}\delta^{(2)}(\boldsymbol{x},\boldsymbol{y})\right) \qquad (\mathrm{B.12})$$

for any traceless matrix A. This leads to the crucial result that P_i and $R_i + Q_i$ commute with the Noether charge of the global SL(2, ℝ) group (5.63), since the right hand side of (B.12) multiplied by $\mathcal{V}(\boldsymbol{y})$ gives a total derivative in $\boldsymbol{y}$.

Assigning special values to A we get

$$\begin{aligned}\{P_i(\boldsymbol{x}),\mathcal{W}(\boldsymbol{y})\}_* &= -\mathrm{i}(P_iY + (R_i+Q_i)Z)\mathcal{V}^{-1}\delta^{(2)}(\boldsymbol{x},\boldsymbol{y}) \\ &\quad + \tfrac{1}{2}Z\mathcal{V}^{-1}(\boldsymbol{y})\partial_i\delta^{(2)}(\boldsymbol{x},\boldsymbol{y}), \\ \{R_i+Q_i(\boldsymbol{x}),\mathcal{W}(\boldsymbol{y})\}_* &= \mathrm{i}(P_iZ^* - P_i^*Z)\mathcal{V}^{-1}\delta^{(2)}(\boldsymbol{x},\boldsymbol{y}) \\ &\quad - \tfrac{1}{2}Y\mathcal{V}^{-1}(\boldsymbol{y})\partial_i\delta^{(2)}(\boldsymbol{x},\boldsymbol{y}),\end{aligned} \qquad (\mathrm{B.13})$$

where ∂_i always acts on the first argument of the δ-function.

Appendix C

Here we will compute the Hamiltonian and diffeomorphism constraints for the $N = 2$ supergravity in section 5. First we define some abbreviations, some of them are defined already in section 5:

$$\begin{aligned}\widehat{P}_\mu &\equiv P_\mu - \overline{\psi}_\mu\chi, \\ \widetilde{P} &\equiv \mathrm{Tr}(\mathcal{W}\mathcal{V}Z) + \epsilon^{ij}\overline{\psi}_i\gamma_j\chi = -n\widehat{P}_n, \\ \widetilde{R} &\equiv -\mathrm{Tr}(\mathcal{W}\mathcal{V}Y) = -nR_n, \\ G^\mu &\equiv -\tfrac{1}{2}\mathrm{i}e^{-1}\epsilon^{\mu\nu\rho}\overline{\psi}_\nu\psi_\rho + \tfrac{3}{2}\mathrm{i}\bar{\chi}\gamma^\mu\chi.\end{aligned} \qquad (\mathrm{C.1})$$

A straightforward calculation now shows that the total Lagrangian (5.34) is

$$\begin{aligned}\mathcal{L} &= eg^{\mu\nu}\left(-\widehat{P}_\mu\widehat{P}_\nu^* + \tfrac{1}{2}(R_\mu - G_\mu)(R_\nu - G_\nu)\right) \\ &\quad + \epsilon^{\mu\nu\rho}\left(\tfrac{1}{4}e_\mu{}^aF_{\nu\rho a} + \overline{\psi}_\mu\widetilde{D}_\nu\psi_\rho + (P_\mu - \tfrac{1}{2}\overline{\psi}_\mu\chi)\bar{\chi}\gamma_\rho\psi_\nu - (P_\mu^* - \tfrac{1}{2}\bar{\chi}\psi_\mu)\overline{\psi}_\nu\gamma_\rho\chi\right) \\ &\quad + \tfrac{1}{2}e\left(\widetilde{D}_\mu\bar{\chi}\gamma^\mu\chi - \bar{\chi}\gamma^\mu\widetilde{D}_\mu\chi - 3\bar{\chi}\chi\,\bar{\chi}\chi\right),\end{aligned} \qquad (\mathrm{C.2})$$

where the derivative $\widetilde{D}_\mu$ is the equal to D_μ, but the gauge field Q_μ replaced by $Q_\mu + R_\mu$, which is equal to $-\frac{1}{2}\mathrm{Tr}(\mathcal{V}^{-1}\partial_\mu\mathcal{V}\,Y)$; thus only the first line in (C.2) depends on Q_μ and the second order Lagrangian [with Q_μ defined by (5.12)] is simply given by dropping the $R_\mu - G_\mu$ terms in (C.2).

In this notation the second class constraint T_i (eq. 5.44) just becomes $T_i = R_i - G_i$. To differentiate the Lagrangian with respect to $e_t{}^a$ we use

$$\frac{\delta}{\delta e_t{}^a}(eg^{\mu\nu})X_\mu Y_\nu = e_a{}^t(eh^{ij}X_iY_j - nX_nY_n) - ne_{ja}h^{ij}(X_nY_i + X_iY_n) \qquad (\mathrm{C.3})$$

and we must remember that G_μ depends on the dreibein:

$$\frac{\delta}{\delta e_t{}^a} G_\mu = \delta_\mu^t e_a{}^\nu G_\nu - e_a{}^t G_\mu + e_{\mu a} G^t + \tfrac{3}{2} \mathrm{i} e^{-1} g_{\mu i} \epsilon_{abc} \epsilon^{ij} e_j{}^b \bar{\chi} \gamma^c \chi \,. \qquad (C.4)$$

This yields

$$\begin{aligned} \mathcal{H}_a &= \delta \mathcal{L} / \delta e_t{}^a \\ &= -e_a{}^t \Big(e h^{ij} \widehat{P}_i \widehat{P}_j^* - n^{-1} (\widetilde{P} \widetilde{P}^* - \tfrac{1}{2} \widetilde{R} \widetilde{R}) - \tfrac{1}{2} n G_n G_n \Big) \\ &\quad - h^{ij} e_{ja} \Big(\widehat{P}_i \widetilde{P}^* + \widetilde{P} \widehat{P}_i^* + R_i (n R_n - n G_n) \Big) \\ &\quad + \epsilon^{ij} \Big(\tfrac{1}{4} F_{ija} + (P_i - \tfrac{1}{2} \overline{\psi}_i \chi) \bar{\chi} \gamma_a \psi_j - (P_i^* - \tfrac{1}{2} \bar{\chi} \psi_i) \overline{\psi}_j \gamma_a \chi \Big) \\ &\quad + e e_{[a}{}^t e_{b]}{}^i \Big(\widetilde{D}_i \bar{\chi} \gamma^b \chi - \bar{\chi} \gamma^b \widetilde{D}_i \chi \Big) - \tfrac{3}{2} e e_a{}^t \bar{\chi} \chi \bar{\chi} \chi \,, \end{aligned} \qquad (C.5)$$

where we dropped terms proportional to $R_i - G_i$. This is the $N = 2$ analog to (4.15) for pure supergravity without matter fields. The generators of diffeomorphisms are formed in a similar way here, i.e., we have to compute

$$\mathcal{D}_k = e_k{}^a \mathcal{H}_a + Q_k T + A_k{}^a L_a + \overline{\psi}_k \mathcal{S} + \overline{\mathcal{S}} \psi_k \,. \qquad (C.6)$$

The calculation simplifies if one rewrites the other constraints in terms of the fields introduced above, which leads to $T = n(G_n - R_n)$ and

$$\mathcal{S} = -\chi \widetilde{P}^* + \epsilon^{ij} \widetilde{D}_i \psi_j - \epsilon^{ij} \gamma_i \chi \widehat{P}_j^* + \tfrac{1}{2} \epsilon^{ij} (\chi \bar{\chi} \gamma_i \psi_j - \gamma_i \chi \bar{\chi} \psi_j) \,. \qquad (C.7)$$

The result is

$$\begin{aligned} \mathcal{D}_k &= \mathrm{Tr}(\mathcal{W} \partial_k \mathcal{V}) - \tfrac{1}{2} \epsilon^{ij} (\partial_i A_j{}^a e_{ka} + A_k{}^a \partial_i e_{ja}) \\ &\quad + \tfrac{1}{2} e (\partial_k \bar{\chi} \gamma^t \chi - \bar{\chi} \gamma^t \partial_k \chi) - \epsilon^{ij} (\partial_i \overline{\psi}_j \psi_k + \overline{\psi}_k \partial_i \psi_j) \,. \end{aligned} \qquad (C.8)$$

This coincides with (5.59), which was constructed such that $\mathcal{D}[\zeta^k] = \int \zeta^k \mathcal{D}_k$ provides the Lie derivative of any field by taking the Dirac bracket $\{\mathcal{D}[\zeta^k], \phi\}_* = \mathcal{L}_\zeta \phi$.

The most natural definition for the Hamiltonian constraint would now be the time component of $\mathcal{H}_a$, i.e. $e_t{}^a \mathcal{H}_a$, but unfortunately this contains the Lagrange multiplier $e_t{}^a$ and, even worse, it is not polynomial in the canonical variables. But we are free to take any linear combination of the $\mathcal{H}_a$'s that is independent of $\mathcal{D}_k$ to be the Hamiltonian constraint. It turns out that we get a polynomial function by taking

$$\begin{aligned} \mathcal{H} &= e e^{ta} \mathcal{H}_a + \tfrac{1}{2} (n R_n + n G_n) T \\ &= \widetilde{P} \widetilde{P}^* - n e h^{ij} \Big(\widehat{P}_i \widehat{P}_j^* - \widetilde{D}_i \bar{\chi} \gamma_j \chi + \bar{\chi} \gamma_i \widetilde{D}_j \chi \Big) \\ &\quad + e \epsilon^{ij} \Big(\tfrac{1}{4} e^{ta} F_{ija} + (P_i - \tfrac{1}{2} \overline{\psi}_i \chi) \bar{\chi} \gamma^t \psi_j - (P_i^* - \tfrac{1}{2} \bar{\chi} \psi_i) \overline{\psi}_j \gamma^t \chi \Big) \\ &\quad - \tfrac{3}{2} e^2 \bar{\chi} \chi \bar{\chi} \chi \,. \end{aligned} \qquad (C.9)$$

The only terms that are not obviously polynomial are $ee^{ta} = -\frac{1}{2}\epsilon^{abc}\epsilon^{ij}e_{ib}e_{jc}$ and $enh^{ij} = \epsilon^{ik}\epsilon^{jl}g_{kl}$. Note that in second order formalism for Q_μ we get the same expression for the Hamiltonian constraint without adding the term proportional to T. As in the four-dimensional theory (eq. 3.14) the polynomialized Hamiltonian constraint is a density of weight 2. The overall multiplication by the dreibein in (C.6) and (C.9) leads to new solutions with degenerate metric, which are not included in the theory defined by the Lagrangian (C.2), because only for a non-degenerate metric do we have $\mathcal{H} = \mathcal{D}_k = 0 \Rightarrow \mathcal{H}_a = 0$.

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